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Uncertainty quantification in European type contingent claims

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I lovingly dedicate this thesis to my family
who supported me each step of the way.
Thank you for your endless support and for
all the sacrifices that you have made for me.

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Abstract

In this report, we study, using real data from the market, the impact of the uncertainty in the volatility on the European options pricing and hedging strategy in the single asset and the multi assets case. This effect was measured and quantified by using a variety of methods mainly Polynomial Chaos, Monte Carlo and sparse grid. These methods provide a means of computing the variables of interest statistics such as the expected value, the variance and the density estimation. Numerical tests show that Polynomial Chaos in the 1D case as well as sparse grid in the multidimensional case yield results with high precision and relatively less computational time than the standard Monte Carlo method.

Key words

European options, uncertainty quantification, volatility, pricing, hedging strategy, Monte Carlo, Polynomial Chaos, sparse grid.

Résumé

Dans ce rapport, nous étudions, en utilisant des données réelles du marché, l'impact de l'incertitude de la volatilité sur la tarification des options européennes ainsi que la stratégie de couverture dans le cas monodimensionnel et multidimensionnel. Cet effet a été mesuré et quantifié en utilisant une variété de méthodes principalement les polynômes de chaos, Monte Carlo et la grille clairsemée. Ces méthodes fournissent un moyen pour le calcul des statistiques des variables d'intérêt tels que l'espérance, la variance et l'estimation de densité. Les tests numériques montrent que la méthode des polynômes de chaos dans le cas monodimensionnel ainsi que la grille clairsemée dans le cas multidimensionnel fournissent des résultats avec une grande précision et un temps de calcul relativement moins en les comparant à la méthode standard de Monte Carlo.

Mots clés

Options européennes, quantification de l'incertitude, volatilité, stratégie de couverture, Monte Carlo, polynômes de chaos, grille clairsemée.

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Chapter 1

Introduction

1.1 Uncertainty Quantification

Mathematical models are used to simulate a wide range of systems and processes in many fields such as engineering, physics, chemistry and finance. These systems are subject to a wide range of uncertainties. The effects of this uncertainty should be followed through the system meticulously enough to permit one to estimate their effects on the intended use of the model usually related to forecast the simulation outputs.

In deterministic modeling, complete knowledge of input parameters is assumed. This leads to simplified and tractable computations and produces simulations of outputs that correspond to specific choices of inputs. However, most physical, biological, economical and financial processes contain uncertainty.

Though the model equations may be deterministic, it may not be possible to rely on a single deterministic simulation because the input data are not precisely known or are known to admit intrinsic variability. Consequently, the inputs are replaced by random variables in the corresponding mathematical models. It is imperative to incorporate uncertainty from the beginning of the simulations.

The impact of uncertainty in data such as parameter values and initial and boundary conditions has been considered much less in classical numerical analysis. The goal of uncertainty quantification is to investigate the impact of such uncertainty in data and subsequently to provide more reliable predictions for practical problems.

Uncertainty quantification in computational science and engineering is a quite new area aiming at developing methods to describe quantitatively the origin and the impact of model uncertainty on outputs from the model. It bridges several disciplines including statistics, numerical analysis, and computational science.

Uncertainties are in general represented in terms of probability density functions or intervals. The uncertainty in the inputs is then propagated through the mathematical model into the uncertainty of the output quantity of interest, the goal of the computation. In a general framework, probability density is unknown most of the time, thus, a model

should be build and tested against real data in order to improve the a priori input probability density assumptions.

In computational models, the quantification of uncertainties consists of three major phases [**?**]:

1. Model calibration: the input parameters are characterized by their probability distributions.
2. Uncertainty propagation: the variability in the input is propagated to the output
3. Output prediction: the outputs are characterized in terms of their statistical properties.

In the model calibration phase, we identify the uncertain parameters we need to take into account. These parameters are known as stochastic input variables or random coordinates. After determining what parameters to consider, we need to describe them by specifying their probability distributions. This step can be realized by using a prior knowledge and/or experimental observations.

In the uncertainty propagation phase, the uncertainty coming from the stochastic inputs are transmitted to the outputs. In general, the uncertainty propagation entails solving a problem usually more complicated than the original deterministic problem. Many methods exist in the literature and we will be discussing them in the next chapter.

In the last phase, we find output prediction where the outputs are characterized in terms of their statistical properties such as mean, variance, higher statistical moments, probability density function, tail probability and confidence interval.

1.2 Generalities on option pricing

An option is a derivative instrument giving the holder the right, but not the obligation to buy or sell (whether it is an option to buy or sell) a certain amount of a financial asset at an agreed date and at a price fixed in advance. Precise description of an option is based on the following characteristics:

- The nature of the option: Call for an option to buy and Put for an option to sale.
- The underlying asset on which is written the option: in practice, it may be a stock, bond, currency etc,...
- The amount: the quantity of underlying assets to buy or sell.
- The maturity: the expiration date which limits the lifetime of the option. If the option is exercised at any time prior to maturity, it is called American option. If the option can be exercised only at maturity, it is called European option.

- The strike: the price, fixed in advance, as the transaction cost in case of exercise of the option.

The option itself has a price, called the premium. When the option is traded on an organized market, the premium is given by the market. In the absence of trading, the problem of calculating the premium arises. And even for an option listed, it may be advantageous to have a formula or model to detect any abnormalities in the market.
The vendor of the option has to answer two main questions:

1. How much should the buyer of the option pay, in other words how to assess at time $t = 0$ the richness available at time T? This is the pricing problem.

2. How the vendor who receives the premium at time $t = 0$ will manage to produce wealth at time T? This is the hedging problem.

Options have become extremely popular and the reasons behind that can be summarized in these two points:

- Options are attractive tools for investors both for speculation and hedging.

- Their price can be determined; therefore their trading can be done with a certain confidence.

In the literature, we could find a variety of models that helps us to answer these two questions. However, the most famous mathematical model in the options market is the Black and Scholes model introduced in the early 1970's [**?**]. The theory developed in the seventies by Black, Scholes and Merton has revolutionized the world of finance and boomed the operating options markets. In fact, they were the first to propose a model leading to an explicit formula for the price of a European Call option on a stock that does not pay dividend and to determine a hedging strategy allowing the option seller to be hedged perfectly.

Since we will be interested mainly in European contingent claims, we will be using the Black and Scholes model in the 1D case and trying to adapt it in the multidimensional case for the pricing and hedging of the European basket option.

1.3 Project framework

1.3.1 Objectives

The aim of the project is to quantify the uncertainty existing in the volatility on the option pricing and hedging strategy in the case of European options. In fact, we start by a simple case study in 1D when we deal with a Call option. In this case, we used Polynomial Chaos as well as Monte Carlo method. The second part of the work is a generalization to the multidimensional case when we tackle the problem for a basket option.

1.3.2 Related work

In the literature, the concept of volatility has been modeled either as a random variable or as a stochastic process. Some of the early works on stochastic volatility models could be found in [**?**], [**?**] and [**?**]. The first paper with efficient results on modeling stochastic volatility is the one written by Stein and Stein [**?**]. The second approach is when the volatility is rather represented as a random variable. In this way, the random variable reproduces the stochastic behavior of the volatility, which usually happens when this parameter is estimated by historical data. The choice of the distribution can be done by fitting the estimated density of the volatility.

In our study, we will rather choose the second approach. The outcome of our work consists on computing the statistics of the option price as well as the hedging strategy which can be obtained approximately by a Monte Carlo simulation.

However, a sufficiently accurate result requires a huge number of simulations. To overcome this issue, we will be using two methods: Polynomial Chaos in 1D and sparse grid in the multidimensional case.

Polynomial Chaos was first introduced by Wiener [**?**] in the case of uncertain inputs with Gaussian distributions. This result was extended by Cameron and Martin [**?**] for arbitrary random fields.

Apart from Monte Carlo, sparse grid is one of the methods that knew a success to overcome the curse of dimensionality. In its basic form, the method was originally implemented by Smoylak [**?**].

1.3.3 Contributions

In this thesis, our contribution resides essentially in the multidimensional case, i.e. quantifying the uncertainty in the covariance matrix by using the Wishart distribution and measuring the effects of this uncertainty on the European basket price and hedging strategy. Another main contribution is using the sparse grid coupled with the dimension reduction technique which enabled us to reduce significantly the computational cost of our simulations.

1.3.4 Structure of the thesis

In Chapter 2, we will start by quantifying the effects of the uncertainty in the case of vanilla options. Then, the work will be extended in Chapter 3 to the multidimensional case where the option is written on several underlying assets. Finally, Chapter 4 contains the numerical tests and the main results of the thesis regarding both the 1D and the multidimensional case.

Chapter 2

Uncertainty quantification in vanilla options

The formula of Black and Scholes contains two sources of uncertainty: the volatility and the risk free rate. Options generally have a short maturity and thus the uncertainty coming from the risk free rate is negligible comparing to the one coming from the volatility parameter.

The uncertainty in the volatility causes uncertainty in the option price. In this chapter, we will try to quantify and understand the effect of the volatility uncertainty on the option price and Delta hedging strategy in the case of European Call option.

In our study, we preferred to start by a simple problem in the 1D case mainly because we wanted to focus on the uncertainty quantification objective and not to be distracted by the option pricing problem. Once we get satisfying results in the 1D case, a generalization to the multidimensional case should be smooth.

2.1 Uncertainty quantification: A review of literature

The uncertainty can be quantified by using deterministic or probabilistic methods. A good review of the literature is given in [**?**]. In this section, we present briefly these methods.

In the deterministic methods, we find essentially the interval analysis and the sensitivity derivatives methods. Monte Carlo, the moment methods and Polynomial Chaos are the common probabilistic methods used in the uncertainty quantification analysis.

2.1.1 Interval analysis

This method is known also by the name of worst case analysis. In fact, we suppose that the model input lies in an interval that presents all its possible values. In this way, the output consists of all the results performed on the values of the model's inputs and therefore it has a bounded value.

This method was introduced in finance via the uncertain volatility model by Avellaneda et al in [**?**]. In fact, one assumes that volatility parameters lies within two values σ_{min} and σ_{max}. By solving a non linear variation of the PDE of Black and Scholes, Avellaneda et al determined the worst case option prices defined as follows:

$$\begin{cases} V_{min}(S,t) = \min\limits_{\sigma_{min}\leq\sigma\leq\sigma_{max}} V(S,t;\sigma) \\ V_{max}(S,t) = \max\limits_{\sigma_{min}\leq\sigma\leq\sigma_{max}} V(S,t;\sigma) \end{cases}$$

Where V is the option price for a given value of σ, t is the current time and S is the price of the underlying at time t.

Using only the interval where the input exists gives little information about the model output. In fact, the method sets all the possible values that the output could take without specifying which value is more likely to be the most occurring.

2.1.2 Sensitivity derivatives

In finance, the sensitivity derivatives method has known a wide success. Actually, this comes essentially from the wide use of the Greeks of an option. These Greeks are various partial derivatives with respect to a parameter of interest.

If the output u depends on a set of independent variables $(x_i)_{i=1,..,n}$ having an error $(\Delta x_i)_{i=1,..,n}$ associated with it, then a deterministic approximation to the output error, Δu is given by:

$$\Delta u \simeq \left[\sum_{i=1}^{n} \left[(\frac{\partial u}{\partial \xi_i})_i\right]^2 \Delta\xi_i^2\right]^{\frac{1}{2}}$$

Being a local operator, partial derivatives gives the local dependence of a model on a parameter rather than the global uncertainty of the model. It allows quantifying how the output variability depends on the model inputs.

The success of this method is due in a great part to its dependence on the mathematical formulation of the model rather than the input data collection. However, the sensitivity derivatives do not offer the possibility to predict the output behavior in response to the uncertainty existing in the input.

2.1.3 Monte Carlo

The basic idea behind the standard Monte Carlo method consists on following the next steps:

1. To sample input random variables from their known or assumed probability density function.
2. To calculate deterministic output for each sampled input value.
3. To determine the statistics of the output's distribution, e.g: mean, variance,etc...

For the statistics calculation, the integrals are approximated by finite sums. In fact, for a given function g, the Monte Carlo method consists on generating M number of realizations of $g(X(s))$ and then approximates the value of $E[g(X(s))]$:

$$E\left[g(X(s))\right] \approx \sum_{i=1}^{M} \frac{g(X(s;\omega_i))}{M}$$

The Monte Carlo method converges to the exact solution as the number of samples goes to the infinity. However, the rate of convergence of this method is relatively slow since it is of the order of $\frac{1}{\sqrt{M}}$.

2.1.4 Moment methods

Suppose that $Y = F(X)$ where F is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ and $X = (x_1, , x_n)$ is a vector of $\mathbb{R}^n$. The method is based on a Taylor expansion of the function F about the input mean μ_X. In general, since the resulting system of equations becomes very complex for higher order, the second order expansion is employed.
For $k = 1, .., m$, the second-order Taylor approximation of F is given by:

$$F_k(X) \approx F_k(\mu_X) + \sum_{i=1}^{n} \frac{\partial F_k(\mu_X)}{\partial x_i}(X_i - \mu_i) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 F_k(\mu_X)}{\partial x_i \partial x_j}(X_i - \mu_i)(X_j - \mu_j)$$

From this approximation, one can compute the output statistics mainly the expected value and the variance as follows:
- The mean approximation:

$$\mu_Y = E[F(X)] \approx \left(F_k(\mu_X) + \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 F_k(\mu_X)}{\partial x_i^2}\sigma_i^2 \right)_{k=1,\ldots,m}$$

- The covariance Approximation:

$$cov(Y_k, Y_l) = E[(F_k(X) - \mu_{Y_k})(F_l(X) - \mu_{Y_l})] \approx \sum_{i=1}^{n} \frac{\partial F_k(\mu_X)}{\partial x_i}\frac{\partial F_l(\mu_X)}{\partial x_i}\sigma_{X_i}^2$$

2.1.5 Polynomial Chaos

Polynomial Chaos (PC) is a method used to represent uncertain input quantities as random variables. The term "Chaos" is related to the uncertainty existing in the input characterization whereas "Polynomial" refers to the use of polynomial expansions to propagate uncertainties.

Any random variable having a finite variance could be modeled by a finite sum of polynomial expansion. The family of polynomials used in the representation satisfies the orthogonality condition with respect to the input probability density function.

The basis of polynomials used to represent the output serves as a tool to enable the computation of its statistics, computed generally using PC coupled with MC. Depending on the solver speed, the PC/MC method is much more faster than crude MC method.

A more detailed presentation of Polynomial Chaos is the subject of the following section of this chapter. At the end of this brief revue of literature, we chose Polynomial

Chaos to represent the uncertainty in the 1D case. This choice is justified mainly for two reasons:

- Our aim to characterize the output of the pricing model guides us to choose a probabilistic rather than deterministic method.
- The PC methods provide the same results for the output statistics as the standard MC method but it is significantly faster.

In our study, we will be using the Polynomial Chaos approach in the case of 1D for pricing and hedging strategy for the European Call option. In the following section, we will be given a detailed presentation on this technique.

2.2 Quantifying the uncertainty in European Call pricing

The first step is to model the parameters as random variables either by supposing that they follow a certain distribution or by measuring the parameter and fitting the data.
We recall that in our work, the uncertain parameter of the model is the volatility.
When using polynomial chaos, one can choose either to use:

- Collocation methods, which are non intrusive methods based on sampling, using interpolation techniques and projecting a set of deterministic simulations, determined using selected sampled parameter sets, onto a polynomial basis.
- Intrusive methods known as Galerkin methods based on a Galerkin projection in order to minimize the error of the truncated expansion and the resulting set of coupled equations can be solved to obtain the expansion coefficients.

In our work, we opted for collocation methods which seem to be better in terms of error versus computational cost; in other words, these methods show better performances than Galerkin methods for moderate error tolerances. Collocation is more interesting since it has similar convergence rate than Galerkin but offers great flexibility when doing a variety of tests, for further details see [**?**].

Therefore, we will use stochastic collocation in the Polynomial Chaos approximation to represent the European Call price.

2.2.1 Basics of Polynomial Chaos

Let V(S,t;σ) be the price of the Call option. We begin by specifying the uncertainty in the input parameter. Let ξ be the uncertain input which will replace σ in the Call price expression. The uncertainty in ξ is determined by its probability density function (pdf) $\rho(x)$.

Our aim is to determine the statistics of the output V when the input ξ varies. For this aim, we shall start by expressing the output V as a series of orthogonal polynomials $(\varphi_k)_{k\in\mathbb{N}}$ with respect to the pdf $\rho(x)$ as follows:

$$V(S,t;\xi) = \sum_{i=0}^{\infty} v_i(S,t)\varphi_i(\xi) \tag{2.1}$$

Since the number of terms of the expansion in 2.1 is infinite and for computational reasons, the price of the option can be approximated by a truncated sum:

$$V(S,t;\xi) \approx \sum_{i=0}^{p} v_i(S,t)\varphi_i(\xi) \tag{2.2}$$

where p denotes the maximum polynomial degree of the series approximation and $(\varphi_k)_{k\in\mathbb{N}}$ is the k^{th} polynomial of the basis having k as a degree.

Definition 1. (Inner product) *Let (Ω,$\mathbb{F}$,$\mathbb{P}$) be a probability space and f and g two measurable and square integrable functions on this probability space.*
The inner product,denoted $\langle .,. \rangle$, is defined as follows :

$$\langle f, g\rangle = \int_{\Omega} f(\omega)g(\omega)\mathrm{d}\mathbb{P} \tag{2.3}$$

When $\Omega = \mathbb{R}$, F the Borel measure and $\mathbb{P}$ has a probability density function ρ, the expression 2.2 becomes:

$$\langle f, g\rangle = \int_{\mathbb{R}} f(\omega)g(\omega)\rho(\omega)\mathrm{d}\omega \tag{2.4}$$

The polynomial basis family $(\varphi_k)_{k\in\mathbb{N}}$ verifies the orthogonality condition, i.e. $\langle\varphi_k,\varphi_j\rangle = \int_{\mathbb{R}} \varphi_k(\xi)\varphi_j(\xi)\rho(\xi)\mathrm{d}\xi = \delta_{kj}$.

As one could remarks, each term of the composition is formed by a deterministic coefficient v_i and a random component $\varphi_k(\xi)$. In fact, when applying the Polynomial Chaos technique to a random variable, we obtain also a random variable.
The expression of the deterministic coefficients can be obtained using the orthogonality condition as follows:

$$v_i = \frac{\langle V, \varphi_i(\xi)\rangle}{\langle\varphi_i(\xi),\varphi_i(\xi)\rangle} = \frac{\int_{\mathbb{R}} V(S,t;\xi)\varphi_i(\xi)\rho(\xi)\mathrm{d}\xi}{\int_{\mathbb{R}}(\varphi_i(\xi))^2\rho(\xi)\mathrm{d}\xi} \tag{2.5}$$

Theorem 1. *Let $X \in L_2(\Omega,\mathbb{F},\mathbb{P})$ be a random variable expanded as a series of Polynomial Chaos:*

$$X = \sum_{i=0}^{\infty} \alpha_i\varphi_i(\xi) \tag{2.6}$$

And we assume that $\varphi_0 = 1$. In this case, we have:

$$\begin{aligned} E[X] &= \alpha_0 \\ Var[X] &= \sum_{i=1}^{\infty} \alpha_i^2 \langle\varphi_i(\xi),\varphi_i(\xi)\rangle \end{aligned} \tag{2.7}$$

Theorem 1 is very useful since it permits to express the statistics of the random variable X in terms of the deterministic coefficients $(\alpha_i)_{i\in\mathbb{N}}$.

2.2.2 Construction of PC basis for a given density

In this section, we illustrate how to construct a basis of polynomials $(\varphi_j)_{j\in\mathbb{N}}$ that are orthogonal with respect to $\omega(x)$, a positive integrable weight function on a finite domain. For the infinite domain case, $\omega(x)$ has to go to zero at $\pm\infty$ faster than any polynomial. The construction is based on a standard three-term relation for orthogonal polynomials.

Theorem 2. *Let $\omega(x)$ be a positive and integrable function defined on the domain $[a,b]$ for which there exists a collection $(\varphi_j)_{j\in\mathbb{N}}$ of orthogonal polynomials with respect to $\omega(x)$ and φ_j has degree j. Then, the $(\varphi_j)_{j\in\mathbb{N}}$ satisfy a three-term recurrence relation given by:*

$$x\varphi_n(x) = d_{n,n+1}\varphi_{n+1}(x) + d_{n,n}\varphi_n(x) + d_{n,n-1}\varphi_{n-1}(x), \quad n\in\mathbb{N}^* \tag{2.8}$$

where:

$$d_{n,j} = \frac{\int_a^b x\varphi_n(x)\varphi_j(x)\omega(x)\mathrm{d}x}{\int_a^b x\varphi_j^2(x)\omega(x)\mathrm{d}x}$$

The construction of the PC family requires to eliminate the dependency of $d_{n,n+1}$ on φ_{n+1}. For this purpose, Gautchi [**?**] elaborated a number of numerical methods for orthogonal polynomials construction. One of these methods is monic orthogonal polynomials, i.e. the coefficient of the highest degree monomial is one.

In this case, $d_{n,n+1}$ is automatically equals to one since both $x\varphi_n$ and φ_{n-1} are monic. Therefore, the three-term relation in 2.8 becomes:

$$\varphi_{n+1}(x) = (x - d_{n,n})\varphi_n(x) - d_{n,n-1}\varphi_{n-1}(x), \quad n \in \mathbb{N}^* \tag{2.9}$$

Now, we could compute φ_{n+1} from φ_n and φ_{n-1}.

For specific density, the PC basis is known. Dongbin Xiu [**?**] gives a summary of the most known density functions and their associated PC basis. This summary is given in table 2.1.

Table 2.1: Correspondence between the underlying random variable and their PC basis

	Distribution	PC basis	Support
Continuous	Gaussian	Hermite	$\mathbb{R}$
	Gamma	Laguerre	$[0, \infty[$
	Beta	Jacobi	$[a, b]$
	Uniform	Legendre	$[a, b]$
Discrete	Poisson	Charlier	$\{0, 1, 2, ...\}$
	Binomial	Krawtchouk	$\{0, 1, 2, ..., N\}$
	Negative binomial	Meixner	$\{0, 1, 2, ...\}$
	Hypergeometric	Hahn	$\{0, 1, 2, ..., N\}$

2.2.3 Choice of the volatility distribution

In our 1D case, we take the assumption that the volatility is lognormally distributed. This particular choice for the volatility density is going to be justified later on in the report.

Definition 2. *X is a lognormal random variable if its logarithm is normally distributed. Let the parameters μ and σ denote respectively the mean and standard deviation of the variable's logarithm, then we have: $X = e^{\mu+\sigma Z}$ with Z a standard normal variable.*
The density function of X in this case is given by:

$$f_X(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{(log(x)-\mu)^2}{2\sigma^2}}, x > 0. \tag{2.10}$$

For the 1D case, we will assume that the volatility has a lognormal distribution with mean μ_{LN} and standard deviation σ_{LN}. This assumption is justified by many reasons:

- The simplicity and feasibility of the lognormal density: It assures the positivity of the uncertainty parameter, i.e. the volatility.

- Previous studies assuming the same model, see [**?**].

- The empirical density of the volatility obtained from the real market data, see figure 4.1.

The statistics of the volatility can be written in terms of its logarithms statistics μ and σ as follows:

$$\begin{aligned} E[X] &= e^{\mu+\frac{1}{2}\sigma^2} \\ Var[X] &= (e^{\sigma^2}-1)e^{2\mu+\sigma^2} \end{aligned} \tag{2.11}$$

Our aim now is to construct a family of PC that is orthogonal with respect to the density of the lognormal random variable σ.

I.C. Simpson [**?**] specifies the three-term relation in the case of lognormal polynomials. In fact, if we assume that μ and σ are respectively the mean and standard deviation of the logarithm of the lognormal random variable then the relation 2.9 used to form the PC basis is given by:

$$p_{n+1} = (a_n + xb_n)p_n - c_n p_{n-1}$$

where:

$$\begin{aligned} a_n &= -\mu[e^{n\sigma^2}(e^{\sigma^2}+1)-1]e^{(2n-1)\frac{\sigma^2}{2}} \\ b_n &= 1 \\ c_n &= \mu^2(e^{n\sigma^2}-1)e^{(3n-1)\sigma^2} \end{aligned}$$

We verify that the polynomial that we constructed respect the orthogonality condition:

$$\langle p_i, p_j\rangle = \int_0^\infty W(x,\mu,\sigma)p_i(x,\mu,\sigma)p_j(x,\mu,\sigma)\mathrm{d}x = H_i(\mu,\sigma)\delta_{ij}$$

where:

$$H_0 = 1; \quad H_i = \mu^{2i}\prod_{j=1}^{i} e^{j\sigma^2-1}e^{i(3i-1)\frac{\sigma^2}{2}}, \quad i \in \mathbb{N}^*.$$

In fact, the first polynomials are given below:

$$\begin{aligned} p_0 &= 1; \\ p_1 &= x - \mu e^{\frac{\sigma^2}{2}} \\ p_2 &= x^2 - \mu e^{\frac{3}{2}\sigma^2}(e^{\sigma^2}+1)x + \mu^2 e^{3\sigma^2} \end{aligned}$$

However, the problem of the lognormal polynomial as Oliver G.Ernst and al. signaled in [**?**] is that these polynomials, unlike the Hermite polynomials for instance, are not dense in the space $L^2(\Omega,\mathbb{F},\mathbb{P})$.

The density of the PC basis ensures that each element of $V \in \Omega$ can be expressed in a unique way as an expansion of series of polynomials that converges to V. One can check the density of the PC basis in the space $L^2(\Omega,\mathbb{F},\mathbb{P})$ via the theorem of M. Riesz.

Theorem 3. *The polynomials $(\varphi_k)_{k\in\mathbb{N}}$ are dense in $L^2(\Omega,\mathbb{F},\mathbb{P})$ if and only if the Hamburger moment problem is uniquely solvable for the distribution of ξ i.e. the distribution function f_ξ of ξ is uniquely defined by the sequence of its moments:*

$$E[X^n] = \int_{\mathbb{R}} x^n f_X(x)\mathrm{d}x$$

A classical example to demonstrate that the lognormal polynomials are not dense is given in [**?**]. In fact, if we take a lognormal random variable ξ and a measurable function $g : \mathbb{R} \mapsto \mathbb{R}$ satisfying the following assumptions:

- g is odd and 1-periodic.
- $\langle g(log(\xi)), g(log(\xi))\rangle < \infty$

For instance, we could take $g(x) = sin(2\pi x)$. Then, for all $k \in \mathbb{N}$ we have:

$$\langle \psi_k(\xi), g(log(\xi))\rangle = \int_0^\infty \psi_k(x)g(log(x))f_\xi(x)\mathrm{d}x = 0.$$

Proof. We have to demonstrate that: $\left\langle \xi^k, g(log(\xi))\right\rangle = \int_0^\infty x^k g(log(x))f_\xi(x)\mathrm{d}x = 0$. $\forall k \in \mathbb{N}$. We consider the following variables change $y = log(x)$ for all $k \in \mathbb{N}$. This change yields:

$$\begin{aligned}
\int_0^\infty x^k \frac{1}{x\sqrt{2\pi}} e^{\frac{-(log(x))^2}{2}} g(log(x))\mathrm{d}x &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ky} e^{\frac{-y^2}{2}} g(y)\mathrm{d}y \\
&= \frac{e^{\frac{k^2}{2}}}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{\frac{-(y-k)^2}{2}} g(y)\mathrm{d}y \\
&= \frac{e^{\frac{k^2}{2}}}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{\frac{-z^2}{2}} g(z+k)\mathrm{d}z \\
&= \frac{e^{\frac{k^2}{2}}}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{\frac{-z^2}{2}} g(z)\mathrm{d}z \\
&= 0
\end{aligned}$$

And therefore, for $\eta = g(log(\xi)) \in L^2(\Omega,\mathbb{F},\mathbb{P})$, $\eta \neq \sum_{i=0}^{\infty} \alpha_i\psi_i(\xi)$. □

To overcome this issue, one could write the lognormal variable as an exponential of a Gaussian and use Hermite polynomials as a PC basis. In this case, the expression of the price becomes:

$$V(S,t;e^Z) \approx \sum_{i=0}^{p} v_i(S,t)\varphi_i(Z) \tag{2.12}$$

where Z is the normal random variable associated to ξ and $(\varphi_i)_{i\in\mathbb{N}}$ are Hermite polynomials.

2.2.4 Quadrature methods

For the integral computation, one should use quadrature rules to compute numerically the value of the integrals. In our case, since the integral support is $\mathbb{R}$, the most adequate quadrature rule is Gauss-Hermite quadrature.

In the mathematical field of numerical analysis, quadrature methods are approximations of the numerical value of an integral. In general, the calculation of the integral is replaced by taking a weighted sum of a number of points in the field integration. The quadrature methods consist on the approximation of the value of the following integral $I = \int_a^b f(x)\omega(x)\mathrm{d}x$ where $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$ and $\omega : [a, b] \mapsto \mathbb{R}_+$ a weighting function.

The integral I is approximated by the finite sum $\sum_{i=1}^{n} \omega_i f(x_i)$ where $(\omega_i)_{i=1,\dots,n}$ are called the coefficients or the weights of the quadrature and $(x_i)_{i=1,\dots,n}$ are the points or the nodes of the quadrature. The nodes are real, distinct, unique and are the roots of orthogonal polynomials for the scalar product defined in 2.4 and n is the number of points used for the integral estimation. The table 2.2 gives some of the main configurations of Gauss quadrature.

Table 2.2: Correspondence between the weight function and their associated polynomials

Support	Weight function	Orthogonal polynomials
$[-1, 1]$	1	Legendre polynomials
$[-1, 1]$	$(1-x)^\alpha(1+x)^\beta, \alpha, \beta > -1$	Jacobi polynomials
$[0, \infty[$	e^{-x}	Laguerre polynomials
$[0, \infty[$	$x^\alpha e^{-x}, \alpha > -1$	Generalized Laguerre polynomials
$\mathbb{R}$	e^{-x^2}	Hermite polynomials

In the rest of this section, we will be focusing on Gauss-Hermite quadrature since it is the one we will be using during our computations. The quadrature rule in this case is the following one:

$$\int_{\mathbb{R}} f(x)\omega(x)\mathrm{d}x = \int_{\mathbb{R}} f(x)e^{-x^2}\mathrm{d}x \approx \sum_{i=1}^{n} \omega_i f(x_i)$$

The nodes $(x_i)_{i=1,\dots,n}$ are calculated as the roots of the n^{th} Hermite polynomial H_n. For the weights, their expression is given by the following formula:

$$\omega_i = \frac{2^{n+1} n! \sqrt{\pi}}{[H_n'(x_i)]^2}$$

The nodes and weights corresponding to a set of points are summarized in table 2.3. Regarding the integration of f over $\mathbb{R}$, we can simply apply the quadrature formula to the function $x \mapsto f(x)e^{x^2}$.

Table 2.3: Nodes and weights of Gauss Hermite quadrature

Number of points	Nodes	Weights
2	0.7071	0.8862
	−0.7071	0.8862
3	1.2247	0.2954
	0	1.1816
	−1.2247	0.2954
4	1.6506	0.0813
	0.5246	0.8049
	−0.5246	0.8049
	−1.6506	0.0813

2.3 Quantifying the uncertainty in European Call hedging

Using the polynomial chaos technique enable us not only to develop simple expressions for the statistics of the price(see theorem 1 above) but also for the Delta hedging parameter. In hedging, Delta is a sensitivity parameter that measures the sensitivity of the option, or more generally that of the portfolio to variations in the price at time t, i.e. $\Delta = \frac{dV}{dS}$. In the case of call option, $\Delta = N(d_1)$ and it varies between 0 and 1. The call option whose underlying does not pay dividends will grow with the Delta value, i.e. Delta tends to 0 when the option is out of money and tend to 1 when the option is in the money. Using a centered difference finite approximation, we could determine the statistics of Delta.
The expectation $E[\Delta]$ is approximated by:

$$\begin{aligned} E\left[\frac{\partial V(S_i;\xi)}{\partial S}\right] &\approx E\left[\frac{V(S_{i+1};\xi) - V(S_{i-1};\xi)}{2h}\right] \\ &\approx \frac{E[V(S_{i+1};\xi)] - E[V(S_{i-1};\xi)]}{2h} \\ &= \frac{\alpha_0(S_{i+1}) - \alpha_0(S_{i-1})}{2h} \end{aligned}$$

where α_0 is the first term in the series expansion in equation 2.6 , $S_i = ih + S_0$ and h denotes the grid spacing. For Delta variance computation, we have:

$$\begin{aligned} Var[\Delta] &= E[\Delta^2] - E[\Delta]^2 \\ &\approx E\left[\frac{(V(S_{i+1};\xi) - V(S_{i-1};\xi))^2}{4h^2}\right] - \left(\frac{\alpha_0(S_{i+1}) - \alpha_0(S_{i-1})}{2h}\right)^2 \end{aligned}$$

$$
\begin{aligned}
& E\left[\frac{(V(S_{i+1};\xi)-V(S_{i-1};\xi))^2}{4h^2}\right] \\
& \approx E\left[\frac{V(S_{i+1};\xi)^2+V(S_{i-1};\xi)^2-2V(S_{i+1};\xi)V(S_{i-1};\xi)}{4h^2}\right] \\
& = \frac{E[V(S_{i+1};\xi)^2]+E[V(S_{i-1};\xi)^2]-2E[V(S_{i+1};\xi)V(S_{i-1};\xi)]}{4h^2} \\
& \approx \frac{1}{4h^2}\sum_{k=0}^{p}\left[\alpha_k(S_{i+1})^2+\alpha_k(S_{i-1})^2-2\alpha_k(S_{i+1})\alpha_k(S_{i-1})\right]\int_{\mathbb{R}}\varphi_k^2(\xi)\rho_(\xi)\mathrm{d}\xi \\
& = \frac{1}{4h^2}\sum_{k=0}^{p}\left[\alpha_k(S_{i+1})-\alpha_k(S_{i-1})\right]^2\langle\varphi_k,\varphi_k\rangle
\end{aligned}
$$

And finally, we get the following approximation for the variance of Δ:

$$Var[\Delta] \approx \frac{1}{4h^2}\sum_{k=1}^{p}\langle\varphi_k,\varphi_k\rangle\left[\alpha_k(S_{i+1})-\alpha_k(S_{i-1})\right]^2$$

Being a random variable, the asset price S presents a potential risk. The owner of the option would like to know how the price of the option is affected by the variation of the underlying price. This measure is nothing but the option Delta.

The goal behind hedging is to assure that the value of the portfolio will stay the same no matter how the asset price varies.
We will consider a portfolio consisting of a Call option and a certain amount of the underlying. Let $\Pi_t = \Delta_t S_t - V_t$ denotes the portfolio value at time t. We should note that if Δ does not depend on time, the value of the portfolio will be Delta-neutral, i.e. $\frac{\mathrm{d}\Pi}{\mathrm{d}S} = 0$. However, this is not the case in the practice; that is why one should follow a hedging strategy to ensure that the value of the portfolio will be risk free.

One of the known hedging strategies is the Delta hedging. As its name indicates, this strategy relies essentially on balancing the value of the portfolio at discrete time using the Delta of the option. In order to make the option risk free, we want to have a long position of Δ units of the underlying and a short position on Call option at all times $t \leqslant T$ so that the portfolio position at time t will be: $\Delta_t S_t - V_t$.

At initial time, we borrow $\Delta_0 S_0$ from a bank account and use it to buy Δ_0 units of the underlying and since we will be receiving initially V_0 for the option price, the portfolio value at $t = t_0$ is $\Pi_0 = \Delta_0 S_0 - V_0$. At this stage, our portfolio is Delta-neutral. However, as time goes, the portfolio value changes and we should adjust the amount of stocks Δ. In fact at $t = t_1$, we hold the portfolio $\Pi_1 = \Delta_0 S_1 - V_1$. We can easily check that at $t = t_1$ we want to have the portfolio $\Pi_1 = \Delta_1 S_1 - V_1$. Therefore, in case $\Delta_1 \neq \Delta_0$, we should balance the portfolio value by buying the quantity $\Delta_1 - \Delta_0$ of the underlying.

As needed, we can use the bank account to borrow or lend money. The portfolio hedging must be rebalanced in the same way throughout all the period at discrete times.

In the limit, as we increase the frequency of rebalancing, this creates a perfect hedge.

One of our study's objectives is to be able to see how the uncertainty in the volatility affects the Delta hedging strategy. To this end,we choose a number of hedging dates N and time steps $\delta t = \frac{T}{N}$ where T represents the maturity of the Call option. We assume that $\beta_0 = f(0; S_0) - \frac{\partial f(0;S_0)}{\partial S_0}$ where $f(t; S_0)$ represents the Call value at time t and β_t is the bank account position value at time t.
At each instant $t = n\delta t$ where $n = 0, 1, ..., N-1$, we follow these steps:

- Compute the value $\Delta(t) = \frac{\partial f(t;S_0)}{\partial S_0}$.
- Generate M paths for the underlying price $S(n\delta t, \omega)$.
- Generate the corresponding time discrete realizations for the processes $\alpha_n = \Delta(n\delta t)$ and β_n and the portfolio value $\Pi_n = \alpha_n S_n + \beta_n B_n$.

Our goal is to generate the portfolio value after settling the contract at time $t = T$: $\Pi_N = \alpha_N S_N + \beta_N B_N$ and to draw the density of the tracking error defined as $e_N = \Pi_N - max(S_N - K, 0)$.
The tracking error measures the difference between the value of hedging strategy designed to replicate the option value and the payoff of this option at maturity; it is therefore an essential quantity if we want to assess the model risk that stems from applying the standard Black-Scholes theory in markets.
A positive (negative) value of e_N indicates that we made a profit(loss) on our hedge.

In the output prediction phase,we need to determine the density of the model outputs (price, Delta, hedging strategy,...) in order to see how the uncertainty is propagated in the model. For that, we will be using the kernel density estimation method.

2.4 Kernel density estimation

2.4.1 Basics

In statistics, kernel estimation or Parzen-Rozenblatt method is a non-parametric method of estimating the probability density of a random variable. It is based on a sample of a statistical population and allows estimating the density at any point of support. In this sense, this method generalizes cleverly the estimation method with a histogram.

Suppose that we have an independent and identically distributed sample $\{x_1, x_2, ..., x_n\}$ of a random variable x, then the kernel density estimator is:

$$\hat{f}_h(x) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{x - x_i}{h}\right)$$

Where K is the kernel, a function satisfying:

- $\int K(x)\mathrm{d}x = 1$.
- $\int xK(x)\mathrm{d}x = 0$.
- $\int x^2K(x)\mathrm{d}x = k_2 < \infty$.

Here h is the bandwidth, a positive parameter which governs the degree of smoothing of the estimate. Often, K is chosen to be the density of a standard Gaussian function:

$$K(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$$

Other examples of kernel functions can be found in table 2.4

Table 2.4: Examples of kernel functions

Kernel	K(t)
Epanechnikov	$\frac{3}{4\sqrt{5}}(1-\frac{1}{5}t^2)$ for $\lvert t\rvert < \sqrt{5}$ 0 otherwise
Biweight	$\frac{15}{16}(1-t^2)^2$ for $\lvert t\rvert < 1$ 0 otherwise
Triangular	$1-\lvert t\rvert$ for $\lvert t\rvert < 1$ 0 otherwise
Rectangular	$\frac{1}{2}$ for $\lvert t\rvert < 1$ 0 otherwise

The kernel method depends on the choice of the two parameters: the kernel function and the bandwidth. Whereas the choice of kernel has little influence on the estimator, the bandwidth choice has great affect on the smoothness of the density estimation. In fact:

- If h decreases to zero, we will under-smooth the density estimation.
- If h increases towards infinity, the density estimation will be over-smoothed.

2.4.2 Optimal bandwidth

Since the bandwidth has a significant impact on the density estimation, it should be chosen optimally in order to minimize the total error in the estimate. The determination of the optimal bandwidth requires the calculation of the accuracy of kernel estimators mainly by determining the mean squared error (MSE). Let $\hat{f}(x)$ be the kernel estimator. The expression of the MSE is given by:

$$MSE(\hat{f}(x)) = E\left[(\hat{f}(x) - f(x))^2\right]$$

In order to measure globally the accuracy of the estimator, we calculate the mean integrated squared error (MISE):

$$\begin{aligned} MISE(\hat{f}) &= \int_{\mathbb{R}} MSE(\hat{f}(x))\mathrm{d}x \\ &= \int_{\mathbb{R}} (E[\hat{f}(x)] - f(x))^2 \mathrm{d}x + \int_{\mathbb{R}} E\left[(\hat{f}(x) - E[\hat{f}(x)])^2\right] \mathrm{d}x \\ &= \int_{\mathbb{R}} Bias^2(\hat{f}(x))\mathrm{d}x + \int_{\mathbb{R}} Var(\hat{f}(x))\mathrm{d}x \end{aligned}$$

We can detect two sources incurred in the estimation which compose the expression of MSE:

- The systematic error: the bias of an estimate.

- The random error: the variance of an estimate.

Now, we will try to develop the expression of each term. We begin by calculating the bias of the estimate.

$$\begin{aligned} E[\hat{f}] &= \frac{1}{n}\sum_{i=1}^{n} \frac{1}{h} E\left[K\left(\frac{x - x_i}{h}\right)\right] \\ &= \frac{1}{n}\sum_{i=1}^{n} \frac{1}{h} \int_{\mathbb{R}} K\left(\frac{x - t}{h}\right) f(t)\mathrm{d}t \\ &= \frac{1}{h} \int_{\mathbb{R}} K\left(\frac{x - t}{h}\right) f(t)\mathrm{d}t \end{aligned}$$

The variable transformation $z = \frac{x-t}{h}$ yields:

$$E[\hat{f}] = \int_{\mathbb{R}} K(z) f(x - hz)\mathrm{d}z \tag{2.13}$$

Expanding $f(x - hz)$ to the second order around x gives:

$$f(x - hz) = f(x) - hzf'(x) + \frac{1}{2}(hz)^2 f''(x) + \circ(h^2). \tag{2.14}$$

Now, we replace the Taylor expansion of $f(x - hz)$ developed in 2.14 in the expression 2.13:

$$\begin{aligned} E[\hat{f}(x)] &= \int_{\mathbb{R}} K(z)f(x)\mathrm{d}z - \int_{\mathbb{R}} K(z)hzf'(x)\mathrm{d}z + \frac{1}{2}\int_{\mathbb{R}} K(z)(hz)^2 f''(x)\mathrm{d}z + \circ(h^2) \\ &= f(x)\int_{\mathbb{R}} K(z)\mathrm{d}z - hf'(x)\int_{\mathbb{R}} zK(z)\mathrm{d}z + \frac{h^2}{2} f"(x)\int_{\mathbb{R}} z^2 K(z)\mathrm{d}z + \circ(h^2) \\ &= f(x) + \frac{h^2}{2} k_2 f''(x) + \circ(h^2) \end{aligned}$$

Since $Bias(\hat{f}(x)) = E[\hat{f}(x)] - f(x)$ then $Bias(\hat{f}(x)) \approx \frac{h^2}{2} k_2 f''(x)$.
We remark that the bias tends to zero as h goes to zero.The bias depends also on the variance of the kernel k_2 and the second derivative of the function f, which represents the curvature of the density at the point x. Now, we turn our attention to the computation of the variance of $\hat{f}(x)$.

$$Var[\hat{f}(x)] = Var\left[\frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x - x_i}{h}\right)\right]$$

Using the fact that $(x_i)_i$ are independent, we get:

$$Var[\hat{f}(x)] = \frac{1}{(nh)^2}\sum_{i=1}^{n} Var\left[K\left(\frac{x - x_i}{h}\right)\right]$$

where:

$$\begin{aligned} Var\left[K\left(\frac{x - x_i}{h}\right)\right] &= E\left[K\left(\frac{x - x_i}{h}\right)^2\right] - \left(E\left[K\left(\frac{x - x_i}{h}\right)\right]^2\right) \\ &= \int_{\mathbb{R}} K\left(\frac{x - t}{h}\right)^2 f(t)\mathrm{d}t - \left(\int_{\mathbb{R}} K\left(\frac{x - t}{h}\right) f(t)\mathrm{d}t\right)^2 \end{aligned}$$

Therefore, the expression of the variance becomes:

$$\begin{aligned} Var[\hat{f}(x)] &= \frac{1}{n}\left(\frac{1}{h^2}\int_{\mathbb{R}} K\left(\frac{x - t}{h}\right)^2 f(t)\mathrm{d}t - \left(\frac{1}{h}\int_{\mathbb{R}} K\left(\frac{x - t}{h}\right) f(t)\mathrm{d}t\right)^2\right) \\ &= \frac{1}{n}\left(\frac{1}{h^2}\int_{\mathbb{R}} K\left(\frac{x - t}{h}\right)^2 f(t)\mathrm{d}t - \left(f(x) + Bias(\hat{f}(x))\right)^2\right) \end{aligned}$$

Substituting $z = \frac{x-t}{h}$ and using a first order Taylor expansion of $f(x - hz)$, one obtains:

$$Var(\hat{f}(x)) = \frac{1}{nh}\int_{\mathbb{R}} K(z)^2(f(x) - hzf'(x) + \circ(h))\mathrm{d}z - \frac{1}{n}\left(f(x) + \circ(h^2)\right)^2$$

The value of the variance can be approximated in the case of large value of n and small h which is the most recurrent case. In fact, the expression of the variance in this case becomes:

$$Var(\hat{f}(x)) \approx \frac{1}{nh} f(x)\int_{\mathbb{R}} K(z)^2\mathrm{d}z$$

We note that this variance decreases as h increases. Let j_2 denotes the value of the integral $\int_{\mathbb{R}} K(z)^2\mathrm{d}z$ then the expression of MISE becomes:

$$MISE(\hat{f}) \approx \frac{1}{4}h^4 k_2^2 \int_{\mathbb{R}} f''(x)^2\mathrm{d}x + \frac{1}{nh} j_2$$

In order to determine the optimal value of the bandwidth h, one should minimize $MISE$. In fact, we calculate the first derivative of $MISE$ with respect to h and we set its value equal to zero in order to find the optimal bandwidth.

$$\frac{\mathrm{d}MISE(\hat{f})}{\mathrm{d}h} = h^3 k_2^2 \int_{\mathbb{R}} f''(x)^2 \mathrm{d}x - \frac{1}{nh^2} j_2 = 0 \tag{2.15}$$

By solving equation 2.15, we find the optimal bandwidth expression:

$$h_{opt} = \left(\frac{j_2}{n k_2^2 \int_{\mathbb{R}} f''(x)^2 \mathrm{d}x} \right)^{\frac{1}{5}}$$

We remark that the value of h_{opt} depends on the kernel that we are going to choose. For instance, if we denote by σ the standard deviation of the sample, the value of h_{opt} when using a Gaussian kernel turns to be equal to:

$$h_{opt} = \left(\frac{4\sigma^5}{3n} \right)^{\frac{1}{5}}$$

Another issue that needs to be dealt with in the density estimation is what we call the boundary effects. The calculations done so far assume that the density support is the whole real line. We should remember that our aim is to estimate the density of the option price and hedging strategy. Since the data set includes points that may be near to zero, the chosen kernel especially if it is the normal one will stray into the region where x is negative.

2.4.3 Boundary effects

In the literature, we could find a variety of methods that enables us to tackle the problem of boundary effects. B.W Silverman discusses in [**?**] a couple of methods for density estimation near the boundary. In this section, we will restraint our presentation to positive random variables.

One way of doing this consists on estimating the density for positive value and setting the estimator equal to zero for negative data. However using kernel density estimation in this way, the density estimated will no longer be a probability density function. In fact, kernel density estimation produces estimates whose integral is equal to the unity. Even if we rescale the estimate so that it looks like a probability density, the weights of the points near the boundary in the integral $\int_0^\infty \hat{f}(x)\mathrm{d}x$ calculation will be undervalued.

We could also try to transform the initial set of data that we have and try to calculate the initial estimator from the transformed one. For instance, Copas and Fryer (1980) [**?**] proposed that we estimate the logarithms of the data. If $\hat{g}$ denotes the estimator of the logarithm of data then our initial estimator $\hat{f}$ is given by $\hat{f}(x) = \frac{1}{x}\hat{g}(log(x))$ for $x > 0$.

The presence of the multiplier $\frac{1}{x}$ could cause some difficulties around the point $x = 0$.

In our representation of the output density, we will be using rather another adaptive method which is the reflection method. It consists on widen the set of initial data by adding reflections of all the boundary points. Let $S = \{X_1, X_2, ..., X_n\}$ be our initial data structure. Using the reflection method, our set will be rather $S' = \{X_1, -X_1, X_2, -X_2, ..., X_n, -X_n\}$. If f^* denotes the estimator of the data S', then the estimator $\hat{f}$ of the original data S can be given by putting:

$$\begin{cases} \hat{f}(x) = 2f^*(x) & \text{if} \quad x \geqslant 0 \\ \qquad = 0 & \text{otherwise} \end{cases}$$

In the usual case, the reflection estimator is set as:

$$\begin{cases} \hat{f}(x) = \dfrac{1}{nh} \displaystyle\sum_{i=1}^{n} \left[K\left(\frac{x - X_i}{h}\right) + K\left(\frac{x + X_i}{h}\right) \right] & \text{if} \quad x \geqslant 0 \\ \qquad = 0 & \text{otherwise} \end{cases}$$

A very useful remark to reduce the computational cost, especially if the set of data is large, is to reflect only points near the boundary and not the whole data set. In fact, if $\frac{X_i}{h}$ is very large, the reflected point $-\frac{X_i}{h}$ will not have a significant effect in the computation of $f^*(x)$ for $x \geqslant 0$. For instance, if K is the normal kernel there is no need to reflect points X_i larger than $4h$.

Chapter 3

Uncertainty quantification in basket options

Basket options are options whose payment at maturity is linked to a portfolio or basket of underlying assets. They have the same characteristics as a standard option with the difference that the underlying is rather a basket of several assets. Therefore, the exercise price is based on the weighted value of the assets in the portfolio.

The underlying basket can be a weighted sum of any asset to the extent that all weights are positive. Thus, we can find basket options on currency, commodity, equity or interest rate. Our focus will be on pricing and hedging basket option on stocks.

Basket options have been developed in order to provide a hedging instrument that would cost less than the standard hedging with several options on each of the assets comprising the basket. Indeed, the premium for the option basket is usually lower than the weighted sum of the cost of individual options. This comparative advantage in cost of hedging is even more important that the correlations between the different components of the basket are low. Thus, the option holder can see the rise of an asset counterbalance, in whole or in part, the fall down of another.

3.1 Approximation methods for basket option pricing

Basket options are much more complicated to price than the standard options since their value depends on the weighted sum of several correlated underlying assets. The main theoretical reason behind this difficulty is that the weighted sum of correlated lognormal random variables does not follow a lognormal distribution.

We assume that there is n underlying assets $(S_i(t))_{i=1,\dots,n}$ that follow the risk-neutral stochastic differential equations:

$$\mathrm{d}S_i(t) = (r - q_i)S_i(t)\mathrm{d}t + \sigma_i S_i(t)\mathrm{d}W_i(t) \tag{3.1}$$

Here r is the risk-free rate, q_i and σ_i are respectively the dividend rate and the volatility of each asset. $(W_i)_i$ are Brownian motions with a correlation structure $(\rho_{ij})_{ij}$ between them.

Let $A(t)$ denotes the weighted arithmetic sum of the underlings, i.e. $A(t) = \sum_{i=1}^{n} \omega_i S_i(t)$. Here, ω_i represents the weight of each asset in the basket. Our interest is in pricing the price of the basket option price. The Cox-Ross fundamental theorem of derivative pricing (1979) [**?**] states that, in a complete market, the no-arbitrage value of the Call basket option will be:

$$V = e^{-rT} E^*[max(A(T) - K, 0)]$$

As one could notice, the value of the basket option is equal to its expected payoff discounted at the risk-free rate where E^* denotes the expected value with respect to the risk-neutral probability density function.
In this section, we develop two types of methods for the valuation of basket Call option:

- The numerical methods mainly the Monte Carlo techniques.
- The analytical approximations.

3.1.1 Monte Carlo

Several authors suggest using the Monte Carlo simulations since they are usually very effective to price multi assets European type contingent claims and can be adapted to any problem of multidimensional integration.

When the random variable has a large variance, we need a large number of simulations to achieve the desired accuracy. Thus, it is interesting to make transformations to reduce the variance of the variable obtained. In our study, we will be mainly combining Monte Carlo with two conventional variance reduction techniques: the control variates and the antithetic variates.

a) Control variates

We want to estimate the integral $I = E[Y]$ and we know a random variable C with mean μ and correlated with Y, this variable is called control variable.
For $\beta > 0$, we set $Y(\beta) = Y - \beta(C - \mu)$, so that we have $E[Y(\beta)] = E[Y] = I$. The variance will be in this case:

$$Var[Y(\beta)] = Var[Y - \beta(C - \mu)] = Var[Y] + \beta^2 Var[C] - 2\beta Cov(Y, C) \tag{3.2}$$

This variance can be less than $Var[Y]$ if β is well chosen. The optimal choice of β is obtained when:

$$\frac{\mathrm{d}Var[Y(\beta)]}{\mathrm{d}\beta} = 0 \Rightarrow \beta_{opt} = \frac{Cov(Y, C)}{Var[C]}$$

In this case, the expression of the new variance could be found by replacing β by β_{opt} in the expression 3.2:

$$Var[Y(\beta_{opt})] = (1 - \rho^2(Y, C))Var[Y] < Var[Y].$$

The method has two difficulties:

- We do not necessarily know a control variable C, that is to say one that has known mean and is correlated with Y.

- Even if we know a control variable C, the optimal value β depends on the covariance between C and Y which is often unknown and therefore must be estimated.

In our case, a control variate candidate is the geometric mean of the assets $C = \left(\prod_{i=1}^{n} S_i\right)^{1/n}$.

On one hand, the distribution of C is lognormal since the assets are lognormally distributed. In fact, $log(C) = \frac{1}{n}\sum_{i=1}^{n} log(Si)$ and we know that the sum of normal distributions is normal. Therefore, $log(C)$ is normally distributed and thus C is lognormally distributed. In this sense, one could easily calculate the statistics of C.

On the other hand, the geometric mean and the arithmetic one in this context are much correlated; therefore, the reduction of variance is more significant in this case.

Since we do not know the values of $Cov(Y, C)$ and $Var[C]$, we have to estimate β_{opt} as follows:

$$\hat{\beta}_{opt} = \frac{\sum_{i=1}^{n} (C_i - \overline{C})(Y_i - \overline{Y})}{\sum_{i=1}^{n} (C_i - \overline{C})^2}.$$

b) Antithetic variates

The basic idea of this technique is to generate N random variables X_i and build N other variables X_i^a having the same probability density but negatively correlated with the generated variables.

Let $\hat{m}_a$ denotes the empirical mean. By applying the antithetic variates technique, one obtains:

$$\hat{m}_a = \frac{1}{2N}\sum_{k=1}^{n} (X_k + X_k^a)$$

$$Cov(X_k, X_j^a) = -\delta_{kj}\sigma_k^2$$

In this case, we will obtain a variance reduction by a $\frac{1}{4n}$ factor. The construction of the new variables X_i^a depends on the distribution of the original sample. For instance, if X

follows a normal distribution with parameters μ and σ then we could take $X^a = 2\mu - X$ as an antithetic variate.

Although the Monte Carlo method is flexible enough to be adapted to the pricing of basket options and allows calculating error bounds, it is relatively long to implement. An alternative to this method is to move towards an analytical approximation of the basket option price.
The key idea of analytical approximations is to replace the unknown distribution of the underlying basket option with a distribution known in the literature and which has attractive features and this by matching the moments of the two distributions. In this report, we will restrict our study to the most known analytic approximations used in the literature.

3.1.2 Beisser approximation

Exploiting the idea of Rogers and Shi (1995) [**?**] for Asian option pricing, Beisser (1999) [**?**] provided an analytical approximation for the basket option price by conditioning on a random variable Z and using Jensen's inequality.

In this case, the price of basket Call option is nothing but the weighted sum of artificial European Call prices:

$$\begin{aligned} E\left[(A(T)-K)^+\right] &= E\left[E\left[(A(T)-K)^+|Z\right]\right] \\ &\geqslant E\left[E\left[(A(T)-K|Z)^+\right]\right] \\ &= E\left[\left(\sum_{i=1}^{n}\omega_i E\left[S_i(T)|Z\right]-K\right)^+\right] \end{aligned}$$

Here, the random variable is $Z = \sum_{i=1}^{n}\omega_i S_i(0)\sigma_i W_i(T)$.

We could note that in this case, all the conditional expectations $E\left[S_i(T)|Z\right]$ are lognormally distributed with respect to one brownian motion $W(T)$. Hence, there exists an x^* such that:

$$\sum_{i=1}^{n}\omega_i E\left[S_i(T)|W(T)=x^*\right] = K$$

By defining:

$$\tilde{K_i} = E\left[S_i(T)|W(T)=x^*\right]$$

The event $\sum_{i=1}^{n}\omega_i E\left[S_i(T)|Z\right] \geqslant K$ is equivalent to $E\left[S_i(T)|Z\right] \geqslant \tilde{K_i}$ for all $i \in \{1,2,...,n\}$.

Using this argument, we conclude:

$$E\left[\left(\sum_{i=1}^{n}\omega_i E\left[S_i(T)|Z\right]-K\right)^+\right]=\sum_{i=1}^{n}\omega_i E\left[\left(E\left[S_i(T)|Z\right]-\tilde{K_i}\right)^+\right]$$
$$=\sum_{i=1}^{n}\omega_i\left[S_i(0)N(d_{1i})-e^{-rT}\tilde{K_i}N(d_{2i})\right]$$

where:

$$\tilde{K_i}=S_i(0)e^{\left(r-\frac{\tilde{\sigma_i}^2}{2}\right)T+\tilde{\sigma_i}\sqrt{T}\phi^{-1}(\mathbb{F}_{S^l})(K)}$$

$$\tilde{\sigma_i}=\frac{\sigma_i\sum_{j=1}^{n}\omega_j\sigma_j\rho_{ij}}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\omega_i\omega_j\rho_{ij}\sigma_i\sigma_j}}$$

$$d_{1i}=\frac{log\left(\frac{S_i(0)}{\tilde{K_i}}\right)+\left(r+\frac{\tilde{\sigma_i}^2}{2}\right)}{\tilde{\sigma_i}\sqrt{T}}\quad\text{and}\quad d_{2i}=d_{1i}-\tilde{\sigma_i}\sqrt{T}\quad\text{for}\quad i=1,...,N$$

Here $S^l=E\left[\sum_{i=1}^{n}S_i|Z\right]$ is a lower bound for $\sum_{i=1}^{n}S_i$. $\mathbb{F}_{S^l}$ is the cumulative distribution function of S^l and $\mathbb{F}_{S^l}(K)$ satisfies the following relation:

$$\sum_{i=1}^{n}\omega_i S_i(0)e^{\left(r-\frac{\tilde{\sigma_i}^2}{2}\right)T+\tilde{\sigma_i}\sqrt{T}\phi^{-1}(\mathbb{F}_{S^l})(K)}=K$$

3.1.3 Lognormal approximation

Gentle (1993) [**?**] was the first to propose an analytical approximation to price basket options. He provided a new method based on the technique of Vorst (1992) [**?**] used for the pricing of Asian options.

The idea is that as the weighted sum of lognormal random variables is unknown, Gentle approximates the arithmetic sum of the assets in the basket by their geometric sum. The geometric sum follows a lognormal distribution and can be easily integrated to calculate the price of the option.

The idea behind the method consists on computing the first two moments of the basket's risk-neutral structure at maturity and then matching them to the lognormal distribution.

To begin with, let's define the pseudo-forward of the basket as $F=\sum_{i=1}^{n}w_iF_i$ where $F_i=S_i(0)e^{(r-q_i)T}$.

Now, we introduce the variable $A^*(T)$ as the ratio of $A(T)$ to F. At this stage, we will express the two first moments of A^* in terms of the problem parameters. To do this, we

need to know the expression of $E[S_i(t)S_j(t)]$. Let's consider the following differential:

$$\mathrm{d}(S_i(t)S_j(t)) = S_i(t)\mathrm{d}S_j(t) + \mathrm{d}S_i(t)S_j(t) + \mathrm{d}S_i(t)\mathrm{d}S_j(t) \tag{3.3}$$

Substituting the expression of $\mathrm{d}S_i(t)$ and $\mathrm{d}S_j(t)$ using the equation 3.1, taking expectation on both sides in 4.12 and recalling that $E[W] = 0$, we conclude that:

$$\mathrm{d}E[(S_i(t)S_j(t))] = (2r - q_i - q_j - \rho_{i,j}\sigma_i\sigma_j)\, E[S_i(t)S_j(t)]\mathrm{d}t$$

which leads through integration to:

$$E[(S_i(t)S_j(t))] = S_i(0)S_j(0)e^{[(2r-q_i-q_j-\rho_{i,j}\sigma_i\sigma_j)t]}$$

Going back to the first two moments of A^*, we obtain easily:

$$\begin{aligned} E[A^*(t)] &= 1 \\ E[(A^*(t))^2] &= \frac{1}{F^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\omega_i\omega_j F_i F_j e^{\sigma_i\sigma_j\rho_{ij}} \end{aligned}$$

Now, we could compute the basket option price simply by using the moments-matched lognormal probability density function. The approximate formula resembles the one of Black Scholes in 1D with some modifications:

$$V_{Basket} = e^{-rT}\left[FN(d_1) - KN(d_2)\right]$$

where:

$$d_1 = \frac{log(F/K) + v/2}{\sqrt{v}}, \quad d_2 = d_1 - \sqrt{v} \quad \text{and} \quad v = log(M_2)$$

Here, $N(.)$ is the standard normal cumulative distribution function.

3.1.4 Reciprocal Gamma approximation

The choice of the inverse gamma distribution to approximate the sum of lognormal variables is justified by the fact that Milevsky and Posner (1998a) [**?**] showed that in the context of Asian options the sum of correlated lognormal random variables converges at the limit to the inverse gamma distribution.

Given that there is some similarity between Asian options and basket options, the authors applied the same technique to price the basket options.

We demonstrate how to value basket options by using the reciprocal gamma distribution as the state price density function for the underlying stochastic variable. This, in turn, allows us to obtain a closed form expression for the price of a basket option, employing moment matching techniques.

Definition 3. Gamma distribution
The density function of a gamma random variable X of parameters (α, β) is given by:

$$g(x;\alpha,\beta) = \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)}, \quad x \geqslant 0$$

where $(\alpha, \beta) \in \mathbb{R}_+^*$ and $\Gamma(.)$ is the Gamma function.

Definition 4. Inverse Gamma distribution
The variable $Y = \frac{1}{X}$ follows the inverse gamma distribution and its density is defined as follows:

$$g_R(y;\alpha,\beta) = \frac{e^{-\frac{1}{y^{\beta}}}}{y^{\alpha+1}\beta^{\alpha}\Gamma(\alpha)} \quad , y > 0$$

The moments of the inverse gamma random variable $Y \sim g_R(\alpha, \beta)$ are given by:

$$E[Y^n] = \frac{1}{\beta^n \prod_{i=1}^{n} (\alpha - i)} \quad , n \in \mathbb{N}^*$$

Taking the two first moments and keeping the same notations as in the previous section, one could write the following system of equations:

$$\begin{cases} \dfrac{1}{\beta(\alpha-1)} = 1 \\ \dfrac{1}{\beta^2(\alpha-1)(\alpha-2)} = M_2 \end{cases} \tag{3.4}$$

Solving the system 3.4 will give us the following value of α and β:

$$\begin{cases} \alpha = \dfrac{2M_2 - 1}{M_2 - 1} \\ \beta = 1 - \dfrac{1}{M_2} \end{cases}$$

Using the inverse gamma distribution with these parameters, we can compute the price of the basket option:

$$V_{Basket} = e^{-rT}\left[FG(\frac{F}{K}, \alpha - 1, \beta) - KG(\frac{F}{K}, \alpha, \beta)\right]$$

Here, $G(.)$ is the cumulative distribution function of the gamma distribution.

3.1.5 Ju's expansion

Based on the idea that the weighted sum of lognormal random variables can be approximated correctly by a lognormal distribution whose first two moments are identical

to those of the true distribution, Ju (2002) [?] develops a more accurate approximation using Taylor expansion.

Indeed, the author considers a Taylor series expansion of the ratio of the characteristic function of the unknown sum of lognormal random variables, compared with the characteristic function of the approximate lognormal random variable around a volatility equal to zero. The underlying assets of the basket generally have volatilities different from zero. In order to develop its Taylor series around zero volatility, Ju assumes a fictitious market where all volatilities are multiplied by a parameter z.
In fact, using the same notations as in 3.1 , the assets' prices are given by:

$$S_i(t) = S_i e^{(r-q_i-\frac{\sigma^2}{2})t+\sigma_i W_i(t)} \quad , i = 1, ..., n$$

whereas in the fictitious market, the prices become:

$$S_i(t) = S_i e^{(r-q_i-z^2\frac{\sigma^2}{2})t+\sigma_i W_i(t)} \quad , i = 1, ..., n$$

When $z = 1$, the author recovers the initial process of the underlying asset. Thus, Taylor expansion around zero volatility means that the development is carried out around $z = 0$ till z^6. This implies that the proposed model converges around small values of z.

Let $A(z) = \sum_{i=1}^{n} \omega_i S_i(z, T)$ be the arithmetic mean with volatilities scaled by the parameter z. Using the moments matching technique, we define the normal random variable $Y(z)$ with mean $m(z)$ and variance $v(z)$ such that the first two moments of $e^{Y(z)}$ match those of $A(z)$.

The adequate parameters can be deducted in the same way as in section 3.1.3 with a minor change by taking $z\sigma_i$ instead of σ_i. In fact, we find the following expressions:

$$m(z) = 2log(U_1) - 0.5log(U_2(z))$$
$$v(z) = log(U_2(z)) - 2log(U_1)$$

where:

$$U_1 = A(0)$$
$$U_2(z) = \sum_{i=1}^{n}\sum_{j=1}^{n} F_i F_j e^{z^2\sigma_i\rho_{ij}\sigma_j T}$$

Let $X(z) = log(A(z))$,then its characteristic function is given as:

$$E\left[e^{i\phi X(z)}\right] = E\left[e^{i\phi Y(z)}\right]\frac{E\left[e^{i\phi X(z)}\right]}{E\left[e^{i\phi Y(z)}\right]} = E\left[e^{i\phi Y(z)}\right] f(z)$$

Whereas we know that $E\left[e^{i\phi Y(z)}\right] = e^{i\phi m(z)-\phi^2\frac{v(z)}{2}}$, Ju performed a Taylor expansion of $f(z)$ up to z^6:

$$f(z) \approx 1 - i\phi d_1(z) - \phi^2 d_2(z) + i\phi^3 d_3(z) + \phi^4 d_4(z) \tag{3.5}$$

where $d_i(z)$ are polynomials of z and terms of higher order then z^6 are ignored. The expression of these polynomials could be found in Appendix I. The expansion 3.5 leads us to approximate the term $E\left[e^{i\phi X(z)}\right]$ as follows:

$$E\left[e^{i\phi X(z)}\right] = e^{i\phi m(z)-\phi^2\frac{v(z)}{2}}\left(1 - i\phi d_1(z) - \phi^2 d_2(z) + i\phi^3 d_3(z) + \phi^4 d_4(z)\right)$$

For the characteristic function expression, an approximation of the density $h(x)$ of $X(1)$ is derived as:

$$h(x) = \left(1 + d_1(1)\frac{\mathrm{d}}{\mathrm{d}x} + d_2(1)\frac{\mathrm{d}^2}{\mathrm{d}x^2}) + d_3(1)\frac{\mathrm{d}^3}{\mathrm{d}x^3}) + d_4(1)\frac{\mathrm{d}^4}{\mathrm{d}x^4})\right)p(x)$$

where $p(x)$ is the normal density with mean $m(1)$ and variance $v(1)$:

$$p(x) = \frac{1}{\sqrt{2\pi v(1)}}e^{-\frac{(x-m(1))^2}{2v(1)}}$$

The approximate price of a basket call is then given by:

$$V_{Basket} = e^{-rT}\left(U_1 N(y_1) - KN(y_2) + K\left(z_1 p(y) + z_2\frac{\mathrm{d}p(y)}{\mathrm{d}y} + z_3\frac{\mathrm{d}^2 p(y)}{\mathrm{d}y^2}\right)\right)$$

where:

$$\begin{aligned} y &= log(K) \\ y_1 &= \frac{m(1)-y}{\sqrt{v(1)}} + \sqrt{v(1)}; \quad y_2 = y_1 - \sqrt{v(1)} \\ z_1 &= d_2(1) - d_3(1) + d_4(1); \quad z_2 = d_3(1) - d_4(1); \quad z_3 = d_4(1) \end{aligned}$$

Note that the first summand is equal to the lognormal approximation and the second summand gives the higher order corrections.

3.2 Sparse grid

The interpolation method called sparse grid technique uses hierarchical bases functions (piecewise linear or polynomial) defined on a set of points of a sparse grid. In fact, the number of points used is reduced compared to the conventional interpolation methods.

This method belongs to the category of sparse approximation methods adapted to large-scale problems. This interpolation uses Smolyak construction. It allows us to select only part of the tensor products that defines the multivariate basic functions. In fact,

we define a hierarchical approach in which the basic functions that have a small support, i.e. a little contribution to the representation of the function to be approximated, are neglected.

The method starts generally from a coarse grid to detect the most important directions and refine these directions. In this section, we will start by presenting the case of full grid then move on to sparse grid case. Our presentation is based essentially on the notes of [**?**].

3.2.1 Full grid

Let's consider the following univariate function $f : [0,1] \mapsto \mathbb{R}$. We recall the sample quadrature rule in 1D as follows:

$$I = \int_0^1 f(x)\omega(x)\mathrm{d}x \approx U_k(f) := \sum_{i=1}^{n_k} \omega_{i,k} f(x_{i,k}) \tag{3.6}$$

The weights $(\omega_i)_i$ and nodes $(x_i)_i$ are determined once the quadrature rule is specified. In our case, since the integration is done over the real line $\mathbb{R}$, we will choose to use Gauss-Hermite quadrature rule.
In the multidimensional case, the first approach is to use the full grid or product method approach to approximate the integral.

Let $I_d = \int_{[0,1]^d} f(x)\mathrm{d}x$ denotes the integral of f over the unit cube $[0,1]^d$. Our aim is to approximate this integral.

The use of full grid consists on a simple product tensor of the univariate quadrature rules. In the general case, if we have d functions $f_1, ..., f_d$ of one real variable, the tensor product is the procedure that builds a function f of d real variables:

$$f = f_1 \otimes ... \otimes f_d$$

where $f(x_1, ..., x_d) = \prod_{i=1}^{d} f_i(x_i)$.

As for the interpolation formula in dimension, it is obtained by applying the tensor product for the approximation formula in Equation 3.6:

$$U_{d,n} f = \sum_{i_1=1}^{n_1} ... \sum_{i_d=1}^{n_d} \omega_{i_1,1} ... \omega_{i_d,d} f(x_{i_1,1}, ..., x_{i_d,d}) \tag{3.7}$$

This classical multidimensional interpolation is defined for nodes of a grid having $\prod_{k=1}^{d} n_k$ points. The implementation of the expression in Equation 3.7 could be computed via the following algorithms:

Algorithm 1 : Computation of product quadrature rule (3.7)

Inputs: $p \in \mathbb{N}$,$i \in \mathbb{N}^d$
Outputs: $P_{d,n}f$
Set $I_f = 0$
Set $p = 1$ and $i = (1, 1, ..., 1)$
while $i \neq (0, 0, ..., 0)$ **do**
  $I_f = I_f + \omega_{i_1,1}...\omega_{i_d,d} f(x_{i_1,1}, ..., x_{i_d,d})$
  Determine the next $i = (i_1, ..., i_d)$ by algorithm 2.
end while

Algorithm 2 : Drop algorithm for the iterative enumeration of all product indices $i = (i_1, ..., i_d)$

Inputs: $p \in \mathbb{N}$,$i \in \mathbb{N}^d$
Outputs: $i \in \mathbb{N}^d$
while $i \neq (0, 0, ..., 0)$ **do**
  $i_p = i_p + 1$
  if $i_p > n_p$ **then**
    if $p = d$ **then**
      return $(0, ..., 0)$
    end if
    $i_p = 1$
    $p = p + 1$
  else
    $p = 1$
    return i
  end if
end while

The algorithm 3.2.1 provides an efficient way to calculate the following vector of indices $i = (i_1, i_2, ..., i_d)$.

The main problem with the full grid approach is that it suffers from the curse of dimensionality. In fact, the number of points used in the grid increases exponentially with the dimension and quickly makes the product method impractical.

To overcome this issue, one could use the sparse grid method instead which enables to overcome to some extent the curse of dimensionality. Table 3.1 summarizes the number of points used in the product method for a 4 points quadrature in the 2D case:

Table 3.1: Number of points of full grid

Dimension	Number of points
4	256
5	1024
10	10^6
20	10^{18}

3.2.2 Sparse Grid

We could define the following difference quadrature for the index $k \in \mathbb{N}$:

$$\Delta_{1,k} = U_{1,k} - U_{1,k-1} \quad \text{with} \quad U_{1,0} = 0 \quad \text{for} \quad k \geqslant 1 \tag{3.8}$$

For a multi-index $k = (k_1, ..., k_d)$, the d-dimensional difference formula is the result of the tensor product of the 1D formula in 3.8:

$$\Delta_{d,k} f = (\Delta_{1,k_1} \otimes \Delta_{1,k_2} \otimes ... \otimes \Delta_{1,k_d}) f \tag{3.9}$$

Now, let's consider a multidimensional function $f : [0,1]^d \mapsto \mathbb{R}$. The integral I_d in this case is nothing but the infinite telescoping sum:

$$I_d = \sum_{k \in \mathbb{N}^d} \Delta_{d,k} f$$

This formula detects the product of all possible combinations of the univariate difference formula. Here $k = (k_1, ..., k_d)$ denotes a multi-index with $k_j > 0$ for $j \in \{1, ..., d\}$ and $\Delta_{d,k} f$ is defined in 3.9.

The sparse grid approach known also as Smoylak algorithm consists on calculating the integral I_d for all the indices $k \in \mathbb{N}^d$ that have their norm $|.|_1$ less than a certain upper bound. To be more explicitly, we consider a level $l \in \mathbb{N}$ and we define the approximation

for the integral I_d for this level as follows:

$$SG_l f = \sum_{|k|_1 \leqslant l+d-1} \Delta_{d,k} f \quad where \quad |k|_1 = \sum_{j=1}^{d} k_j \tag{3.10}$$

We remark that the expression 3.10 is very similar to the one of the product method with a minor difference: the norm used in the product method approach is $|.|_\infty$ whereas in the sparse grid, the norm is rather $|.|_1$.
Keeping in the sum only tensor products corresponding to the indices satisfying $|i|_1 \leqslant l + d - 1$, the Smoylak algorithm eliminates the tensor products of the basis functions having the smaller supports and thus the lowest contributions to the integral calculation.
A numerical example to illustrate Smoylak algorithm could be find in appendix II.

As we could observe Smoylak algorithm is a sort of sum of low order grids. Therefore, the method uses much fewer points. The following figure illustrates this statement in the 2D case for the level $l = 4$:

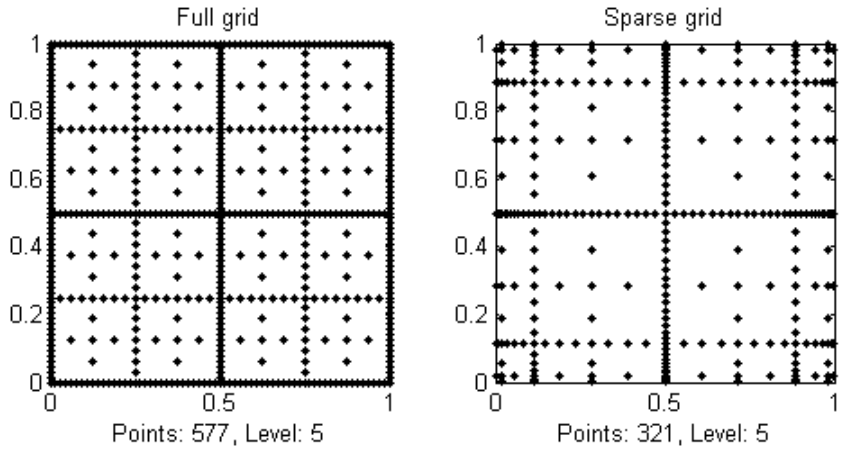


Figure 3.1: Comparison between the number of points in full and sparse grid

In this case, while the product rule uses 577 points, the Smoylak algorithm employs 321 points. This difference becomes more obvious for large dimension. For instance, for the 3D case and level $l = 5$, the product rule uses 3713 points whereas the Smoylak algorithm employs only 1023 points.

The expression 3.10 could be written in a different way in order to facilitate the implementation of algorithms 3 and 4:

$$SG_l f = \sum_{l=d}^{l+d-1} \sum_{|k|_1 = l} \Delta_{d,k} f \tag{3.11}$$

Algorithm 3 : Implementation of the sparse grid method (3.11)

Inputs: $p \in \mathbb{N}$,$k, \hat{k} \in \mathbb{N}^d$
Outputs: $SG_l f$
Set $SG_l f = 0$
Set $m = l + d - 1$
for $l = d, ..., m$ **do**
  Set $p = 1$, $k = (m, 1, ..., 1)$ and $\hat{k} = (m, m, ..., m)$
  while $k \neq (0, 0, ..., 0)$ **do**
    Compute the product formula $\Delta_{d,k}$ using algorithm 3.2.1.
    $SG_l f = 0 = SG_l f + \Delta_{d,k}$
    Determine the next index k using algorithm 4
  end while
end for
return $SG_l f$

Algorithm 4 : Drop algorithm for the iterative enumeration of all product indices $k \in \mathbb{N}^d$, $k_j > 0$ with $|k|_1 = l$

Inputs: $p \in \mathbb{N}$,$k, \hat{k} \in \mathbb{N}^d$
Outputs: $k \in \mathbb{N}^d$
while $k \neq (0, 0, ..., 0)$ **do**
  $k_p = k_p + 1$
  if $k_p > \hat{k}_p$ **then**
    if $p = d$ **then**
      return $(0, ..., 0)$
    end if
    $k_p = 1$
    $p = p + 1$
  else
    for $j = 1, .., p - 1$ **do**
      $\hat{k}_j = \hat{k}_p - k_j + 1$
    end for
    $k_1 = \hat{k}_1$
    $p = 1$
    return k
  end if
end while

We recall that the univariate quadrature rule used here is also the Gauss Hermite quadrature since the domain of integration is the whole real line $\mathbb{R}$.

To illustrate the advantage of sparse grid on Monte Carlo method, let's have a look at the rate of convergence of both methods:

- The convergence rate of MC is $\frac{1}{\sqrt{n}}$. It is independent of the dimension d, but relatively low and a huge number n of function evaluations is needed to achieve a high accuracy.

- The convergence rate of the sparse grid in its simplest form is $n^{-r}(log(n))^{(d-1)(r+1)}$ where r is the degree of smoothness of the integrand.

Appendix III provides some numerical tests using the sparse grid method.

3.3 Uncertainty quantification in pricing

In the multidimensional case, the uncertainty lies in the covariance matrix. The work done in the 1D case is generalized with some modifications. In fact, our approach will consists essentially on calculating the following multidimensional integral:

$$\begin{aligned} E[F(S_T)] &= E[F \circ h(W_T, \Sigma)] \\ &= \int_{\mathbb{R}^K} \int_{\mathbb{R}^d} F \circ h(W_T, \Sigma)\rho(W)\mathrm{d}W\,\rho(\Sigma)\mathrm{d}\Sigma \end{aligned} \tag{3.12}$$

Here F represents the function that expresses the payoff of the basket option in function of the vector $S_T = (S_{T,1}, S_{T,2}, ..., S_{T,d})$ where d is the number of assets. In our case, $F(S_T) = \max(\sum_{i=1}^{d} S_{T,i} - K, 0)$. h is the function that expresses the vector S_T in function of the brownian motions vector W_T and the covariance matrix Σ.
The equation 3.12 could be written in another form:

$$E[F(S_T)] = \int_{\mathbb{R}^K} G(\Sigma)\rho(\Sigma)\mathrm{d}\Sigma \tag{3.13}$$

Here, $G(\Sigma)$ could be seen as the approximation of the basket option price and the integral could be calculated using the sparse grid technique explained in the previous section.

Before computing the value of the price of the basket represented by the integral expression in equation 3.13, we should model the uncertainty in the covariance matrix. Another issue to be taken into consideration is the dimension of the problem. In fact, the integration parameter K is of the order of m^2 where m is the size of the covariance matrix. To overcome this problem, we shall present a technique that enables us to reduce K and thus the computational cost.

a) Modeling the uncertainty

The parameter of uncertainty in the multidimensional case is the covariance matrix. One way of representing the uncertainty in the covariance matrix is to use the Wishart distribution.

Definition 5. Wishart Distribution
Let W be a $K \times K$ absolutely continuous random real matrix whose support R_W,the set of all symmetric positive definite real matrices. Let Σ be a symmetric positive definite matrix and $n > K-1$. We say that W follows a Wishart distribution with parameters Σ and n if its joint probability density function is:

$$f_W(\omega) = \frac{[det(\omega)]^{\frac{n-k-1}{2}} e^{-\frac{1}{2}tr(\Sigma^{-1}\omega)}}{2^{\frac{nK}{2}}[det(\Sigma)]^{\frac{n}{2}}\pi^{\frac{K(K-1)}{4}}\prod_{j=1}^{K}\Gamma(\frac{n+1-j}{2})}$$

where $\Gamma(.)$ is the Gamma function.

The Wishart distribution is a family of probability distributions on symmetrical positive definite matrices, which is the case of the covariance matrix. So, a Wishart random variable is a random matrix. Two parameters characterize this distribution:

- Σ: a symmetric and positive definite matrix.

- n: the degree of freedom.

The first parameter of this distribution represents in our study the covariance matrix, i.e. the parameter which we want to represent as an uncertain quantity. The degree of freedom depends essentially on the set of data used in the generation of the estimated covariance matrix.
The statistics of the matrix W could be written in function of the ones of Σ:

- The expected value of W: $E[W] = n\Sigma$.

- The covariance of W: $Cov[W] = 2n\Sigma \otimes \Sigma$.

Here, $\otimes$ is the usual Kronecker product of matrices.
When generating a random matrix X using the Wishart distribution, the first parameter, i.e. the symmetric and positive definite matrix is taken as the estimated covariance matrix divided by the degree of freedom. The statistics of X are in this case:

$$\begin{cases} E[X] = \Sigma \\ Cov[X] = \dfrac{2}{n}\Sigma \otimes \Sigma \end{cases} \tag{3.14}$$

b)Dimension reduction technique

In this section, we will try to present a technique to reduce the dimension of integration. Before applying this method, we recall that the integration dimension was of the order of m^2 where m is the size of the covariance matrix.

The basic idea behind this technique is to apply the spectral decomposition to the covariance matrix in order to determine its eigenvalues densities by following the next steps:

- Using Wishart distribution, we generate a sample of random matrices having Σ and n as parameters. This step enables us to incorporate uncertainty in the model uncertain input, i.e. the covariance matrix.
- For each single matrix of the previous sample, we perform a spectral decomposition and we store the corresponding eigenvalues.
- Using kernel density estimation technique and the stored eigenvalues, we estimate the density of each eigenvalue.
- We approximate the density estimation using a parametric density function.

In this way, each time we need to calculate the integral, the covariance matrix will be calculated using the spectral decomposition and taking into consideration the estimated densities of the eigenvalues.

Using this technique enabled us to reduce significantly the integration dimension and therefore to minimize the computational cost. In fact, the integration dimension dropped from m^2 to m.

For the density approximation and as this will be shown later on in the numerical tests chapter, we approximate the eigenvalues densities using the normal distribution.

c)Price computation

In the calculation of the integral 3.12, we will follow essentially these approaches:

- Calculating the expression 3.12 via Monte Carlo using the variance reduction techniques.
- Computing the formula 3.13 by using the analytical approximation to calculate $G(\Sigma)$ and then use either sparse grid or Monte Carlo methods.
- Using Moments method based on the Taylor Young expansion.

Since the two first approaches have been detailed in previous sections, we will be concentrating in this section on the third technique.

For a 1D function f, the Taylor Young expansion around the point x_0 gives:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + ... + \frac{f^{(n)}}{n!}(x - x_0)^n + \circ((x - x_0)^{n+1}) \tag{3.15}$$

In our case, we have $V = \psi(\Sigma)$ where V is the basket option price and Σ is the covariance matrix. Let Σ be a random matrix, i.e. $\Sigma_{ij} = \overline{\Sigma}_{ij} + \xi_{ij}$ where $\overline{\Sigma}_{ij}$ is the deterministic component of Σ_{ij} and ξ_{ij} is a Gaussian random variable. The expression in equation 3.15 can be adapted to the case of scalar function of matrix with a few modifications.
In fact, the Taylor expansion of ψ around the point $\overline{\Sigma}$ up to the second order is given by:

$$\psi(\Sigma) = \psi(\overline{\Sigma}) + \sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial\psi(\overline{\Sigma})}{\partial\Sigma_{ij}}\xi_{ij} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m}\frac{\partial^2\psi(\overline{\Sigma})}{\partial\Sigma_{ij}\partial\Sigma_{kl}}\xi_{ij}\xi_{kl} \tag{3.16}$$

Here we should note that $\overline{\Sigma}$ is nothing but the expected value of Σ. As it will be shown later on in the numerical tests section, the price distribution is very close to a normal one. Knowing that the terms beyond the third order of the Taylor expansion are negligible, a second order expansion is sufficient.

The main aim behind using the moments method is to reduce the computational time compared to Monte Carlo and sparse grid simulations. Although the method may not yield good results in the general case. In our study, the results were quite satisfying.

Knowing the probability density function of the price is almost a Gaussian, one could exploit the expansion in equation 3.16 in order to calculate the parameter of this density:

$$E[\psi(\Sigma)] = \psi(\overline{\Sigma}) + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m}\frac{\partial^2\psi(\overline{\Sigma})}{\partial\Sigma_{ij}\partial\Sigma_{kl}}Cov(\xi_{ij}, \xi_{kl})$$

$$Var[\psi(\Sigma)] = \sum_{i=1}^{m}\sum_{j=1}^{m}\left(\frac{\partial\psi(\overline{\Sigma})}{\partial\Sigma_{ij}}\right)^2 Var(\xi_{ij})$$

Using the results of the system 3.14, we have:

$$Cov(\xi_{ij}, \xi_{kl}) = \frac{\Sigma_{ij}\Sigma_{kl} + \Sigma_{ik}\Sigma_{jl}}{n}$$

$$Var[\xi_{ij}] = \frac{\Sigma_{ij}^2 + \Sigma_{ii}\Sigma_{jj}}{n}$$

For the derivatives calculation, we will be using the finite difference method:

$$\frac{\partial\psi(\overline{\Sigma})}{\partial\Sigma_{ij}} = \frac{\psi(\overline{\Sigma} + I_h) - \psi(\overline{\Sigma} - I_h)}{2h}$$

$$\frac{\partial^2\psi(\overline{\Sigma})}{\partial\Sigma_{ij}\partial\Sigma_{kl}} = \frac{\psi(\overline{\Sigma} + I_{h,p}) - \psi(\overline{\Sigma} + I_{h,-p}) - \psi(\overline{\Sigma} + I_{-h,p}) + \psi(\overline{\Sigma} + I_{-h,-p})}{4hp}$$

where h and p are two infinitesimal constants and I_h is the matrix having null values in all its elements except for the element $\Sigma_{ij} = h$. The matrix $I_{x,y}$ is the matrix with null values in all its elements except for the elements $\Sigma_{ij} = x$ and $\Sigma_{kl} = y$.

Using the above equations, one can calculate the expected value and the variance of the price option density. Taking into consideration the results of Monte Carlo and sparse grid and knowing that the density is almost Gaussian, we could plot the price option density as a normal one with the corresponding parameters found with the Taylor Young expansion.

3.4 Uncertainty quantification in hedging

In this section, we will try to quantify the effect of randomness in the covariance matrix on the hedging more precisely on the Delta.

From a practical point of view, the market makers use the Delta as a hedging ratio to create an almost riskless portfolio to meet the nature of their activity. In a Delta-neutral portfolio, any gain on the value of shares as a result of an increase in the price of the underlying asset is offset by the loss on the value of sold Call options and vice versa. In another words, this strategy consists in balancing the number of underlying assets to own in order to hedge the variability of the option with respect to its underlying.

As in the 1D case, the Delta of a basket option has the same expression with a slight modification. In fact, in the basket option case, the Delta is a vector rather than a scalar. However, its components have the same definition as the 1D case where they represent the derivative of the option price with respect to the i^{th} stock price.

Using a centered difference finite approximation combined with Monte Carlo techniques, we were able to calculate the statistics of the hedging parameter Delta and represent its density. For instance, we have:

$$\frac{\partial V}{\partial S_i} = \frac{V(S_i + h) - V(S_i - h)}{2h}$$

Our interest when it comes to hedging is to see the impact of the uncertainty in the covariance matrix on the Delta hedging strategy. We will follow a similar approach to the one detailed in section 3 of the chapter 2 with minor modifications.

In fact, both the underlying price and the Delta of the basket are no longer scalar but rather a vector. In this case, the path generation is quite different from the 1D case since we will need to take into consideration the correlation between the assets. This could be done using a Cholesky factorization for the correlation matrix of the basket option.

Apart from these tiny modifications, the approach remains the same and our aim behind the Delta hedging strategy is always to assure a free-risk portfolio against the variations of the assets prices.

Chapter 4

Numerical tests

4.1 Uncertainty quantification in vanilla options

In this section, we will try to expose and comment the main results that we obtained after applying the Polynomial Chaos to the European Call option. We will try to compute the statistics of the option price as well as those of the Delta hedging strategy.

4.1.1 Data presentation

To begin with, we shall present the data sample that we used to estimate the volatility density. For the volatility computation, we were not able to calculate the implied volatility since we did not have access to the option prices. However, we used historical data from finance.yahoo.com for a study period going from 03 January 2000 till 31 December 2012. The asset in question is the Volkswagen stock "VOW.DE" and we used a 10 days moving window.

Clearly, we can notice from the figure 4.1 that the value of σ presents a certain uncertainty. In fact, this value lies between a minimum $\sigma_{min} = 0.075$ and a maximum $\sigma_{max} = 6.52$. For the statistics of σ, the volatility presents an average $\overline{\sigma} = 36.4\%$ and a standard deviation of $std(\sigma) = 41.2\%$.

In this way, our purpose for the quantification of the effect of the uncertainty in the volatility on the Call option price and Delta hedging strategy is legitimate. In a probabilistic framework, the Polynomial Chaos consists on representing the volatility as a random variable. Therefore, we should look for a distribution to represent the uncertainty in the input.

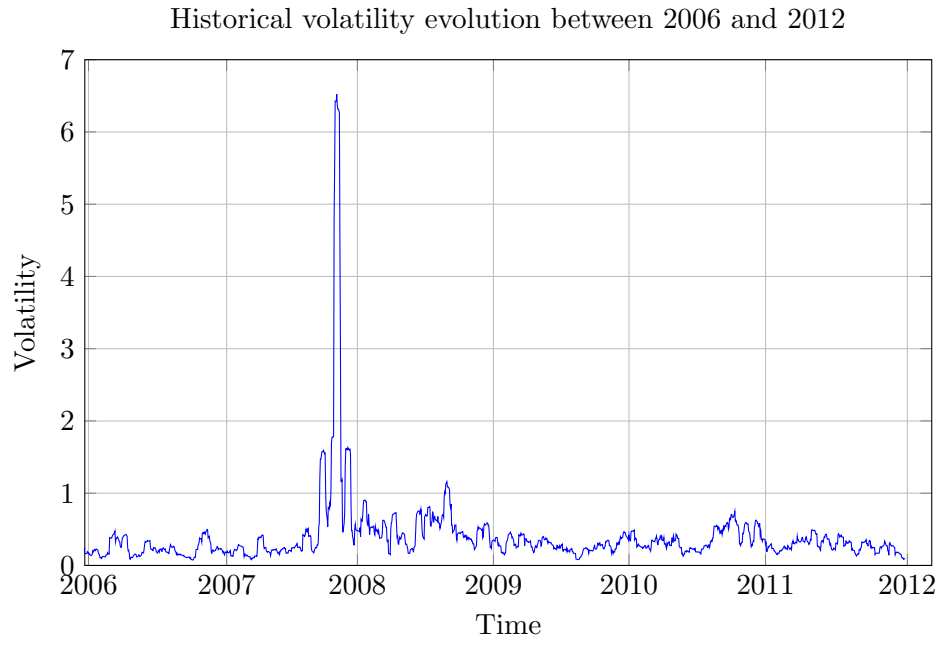


Figure 4.1: Historical volatility from 2006 to 2012

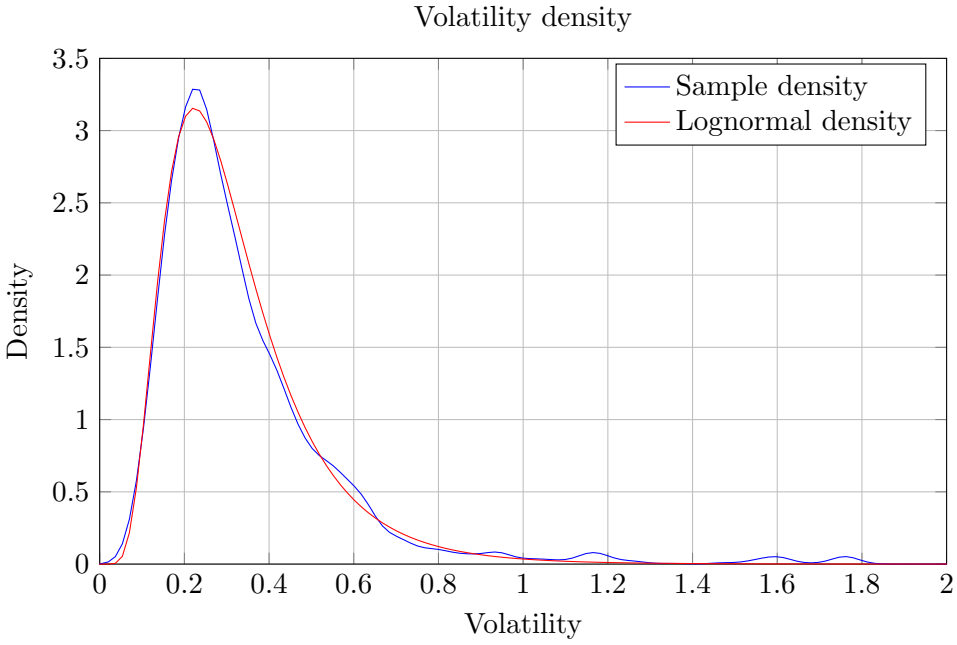


Figure 4.2: Fitting the sample density using lognormal density

Figure 4.2 establishes a comparison between the sample density estimated using the kernel density estimation and the lognormal fitted density that we have chosen. As it can be seen in this figure, the lognormal distribution models quite well the data reasonably although it may fail to capture some rare events especially in the tail.

As argued in chapter 1 particularly in section 2, a variety of reasons pushed us to use the lognormal distribution in representing the volatility.

The sample density in figure 4.2 is plotted using the kernel density estimation technique presented in section 4 of the chapter 2. The parameters of the fitted lognormal distribution are $\mu_{LN} = -1.25$ and $\sigma_{LN} = 0.5$.

4.1.2 Uncertainty quantification in the Call option pricing

In this section, we will try to compare the results of simulation for both standard MC and PC/MC for the pricing of European Call option.

First of all, we will begin by using the standard Monte Carlo. The approach as described in previous section consists on generating a sample containing N values of the volatility $(\sigma_i)_{i=1,..,N}$. This sample is drawn from the estimated distribution.

At this stage, we will compute $V(S,t;\sigma_i)$ for each σ_i so that we obtain the sequence $(V_i(S,t))_{i=1,..,N}$ where $V_i(S,t) = V(S,t;\sigma_i)$. To calculate the value $V_i(S,t)$, one should either solve the Black-Scholes PDE:

$$\frac{\partial V_i}{\partial t} = -\frac{1}{2}S^2\sigma_i^2\frac{\partial^2 V_i}{\partial S^2} - rS\frac{\partial V_i}{\partial S} + rV_i$$

Or we can simply use the Black-Scholes formula to compute the value of the price. Using a sample of $N = 100000$, a risk free rate $r = 5\%$ and a maturity $T = 1$, we had computed, for different stock price, the price, the standard error and 5% confidence interval which are summarized in the table 4.1 for an in-the-money option, an at-the-money option and an out-of-the money option.

Table 4.1: Standard Monte Carlo simulations

Stock price	Call price	Standard error	Confidence interval
100	5.449	0.0561	[5.435, 5.559]
140	23.565	0.0319	[23.318, 23.812]
180	53.24	0.0119	[52.862, 53.618]

Now, we move to the Polynomial Chaos approach. Using the theoretical concept detailed in chapter 2, we will be focusing on computing the statistics of the European Call option.

All along this first part, we have chosen the highest polynomial degree in the PC expansion to be $p = 30$ and the order of the Gauss-Hermite quadrature rule used to approximate the value of the integrals to be $J = 50$. Taking these parameters ensures an error of 10^{-5} in the computed result of the equation 2.12.

Using polynomial Chaos, we compute the mean and standard deviation of the Call price for different values of the underlying price. These results can be found in table 4.2.

Figures 4.3 and 4.4 show how the mean and the standard deviation of $V(S,t;\sigma)$ vary in general with S at the initial time $t = 0$.

Table 4.2: Polynomial Chaos simulations

Stock price	Call price	Standard deviation
100	5.685	5.688
140	23.404	9.012
180	52.9768	7.459

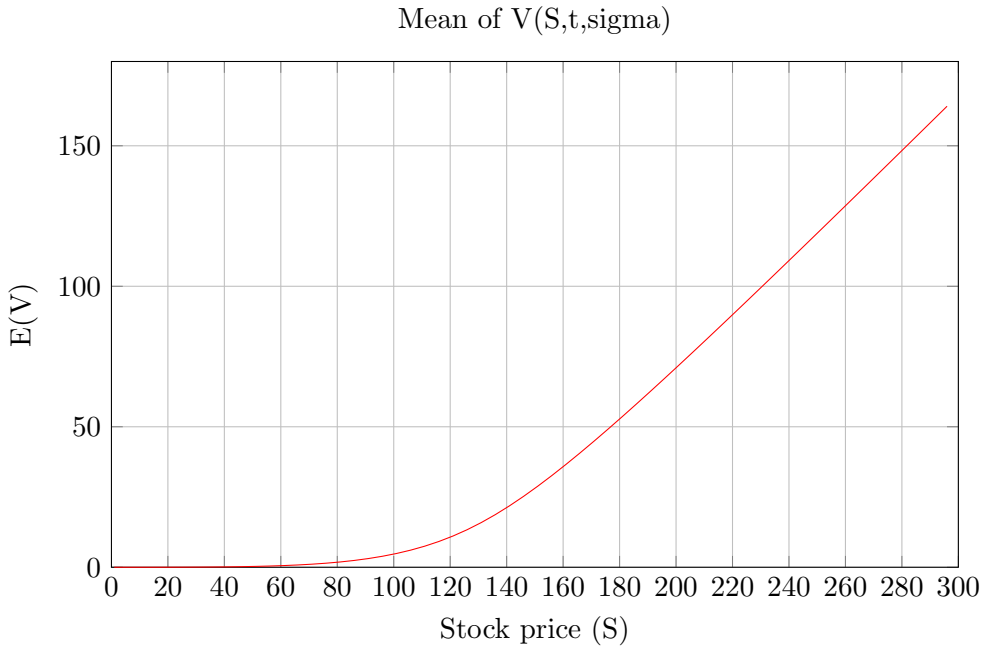


Figure 4.3: The mean of $V(S,t;\sigma)$

The sample density of the Call price may be computed using the kernel density estimation method. The option price densities for $S = 100, 140, 180$ are given in figure 4.5.

The figures (a) and (c) in 4.5 show that for both in or out-of-the money options, the Call price distribution is highly skewed whereas figure (b), which represents at-the-money option density, indicates that the Call option density has a similar shape to the input density.

For the in or out-of-the-money Call options, we note that the density tends to place a large amount of mass near the minimum option value for the given S. This could be explained by the fact that for small values of σ, the option price remains almost the same.

The figure 4.6 compares the mean and the standard deviation found using both a standard MC and the PC/MC methods. The figures show that the two methods yield the same results.

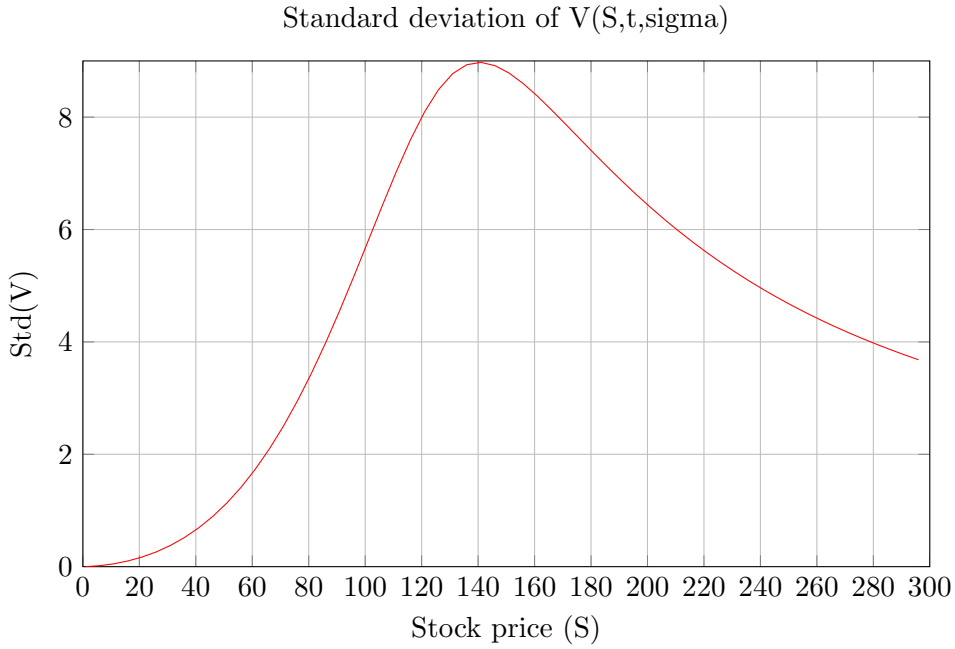


Figure 4.4: The standard deviation of $V(S,t;\sigma)$

4.1.3 Uncertainty quantification in the Call option hedging

In this section, we will try in the first place to measure the effect of the volatility uncertainty on the Delta of the Call option and then to see how it affects the Delta hedging strategy.

Figures 4.7 and 4.8 show the mean and the standard deviation for the Delta of a European Call.

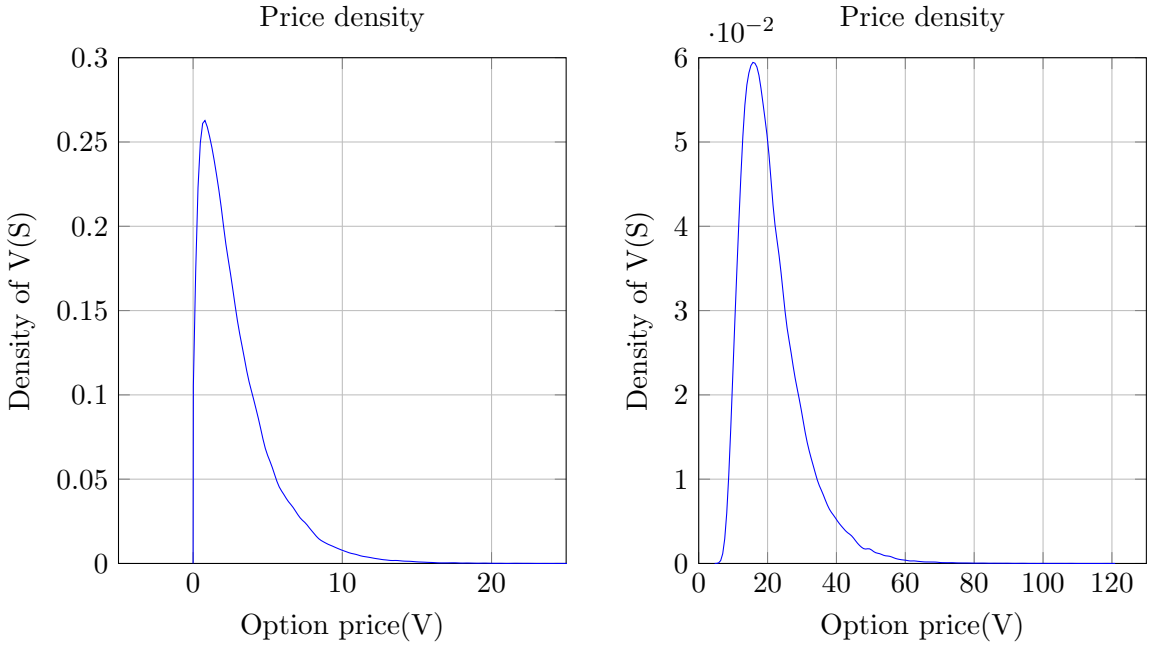
As we already mentioned before, the Call Delta is an increasing function of the underlying price and its value lies in the interval $[0,1]$. In fact, we could distinguish three parts in the figure 4.7:

- Out-of-the-money options: the Delta is between 0 and 0.5. More the Delta is close to zero, the option is less sensitive to changes in the underlying.

- At-the-money options: the Delta is close to 0.5.

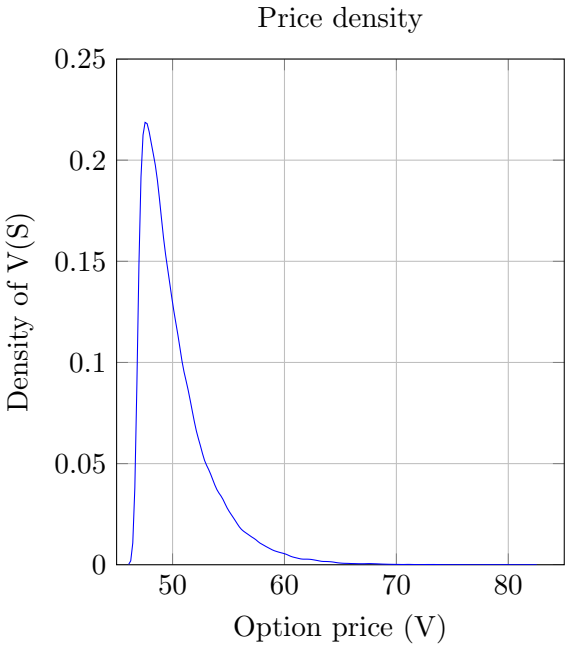
- In-the-money options: the Delta is between 0.5 and 1. More delta approaches 1, the higher the premium option replicates the fluctuations in the underlying.

In figure 4.8, we observe that for at-the-money options, the Delta is insensitive to the uncertainty in the volatility while it is more sensitive to this uncertainty for moderately in or out-of-the-money options. This remark is confirmed when we represent the Delta densities in figure 4.9.



(a) $V(100, 0; \sigma)$

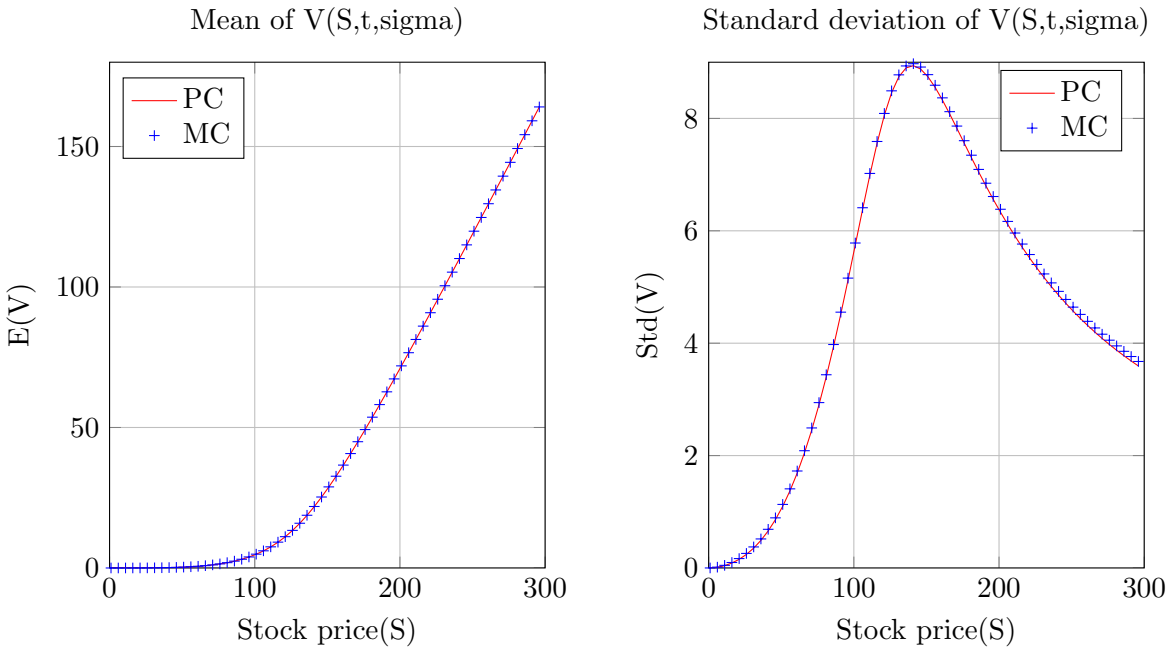
(b) $V(140, 0; \sigma)$



(c) $V(180, 0; \sigma)$

Figure 4.5: Call option price density

The Delta is of particular importance in the theory of derivatives as it enables the setting of a hedging strategy that makes the portfolio risk-free towards the underlying price fluctuations. So, how the uncertainty in volatility affects the Delta hedging strategy? Figure 4.10 presents the tracking error density after settling the contract for different time steps and using the same characteristics for the Call option and $M = 10000$ for the generation of the paths of the underlying price.



(a) Call price mean

(b) Call price standard deviation

Figure 4.6: Comparison between MC and PC

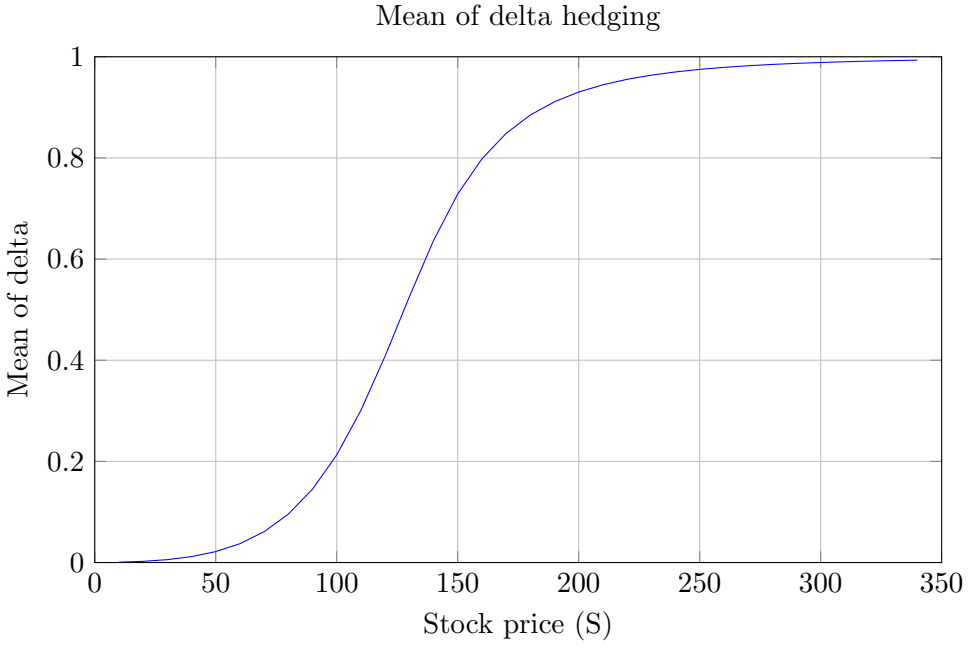


Figure 4.7: The mean of the Call Delta

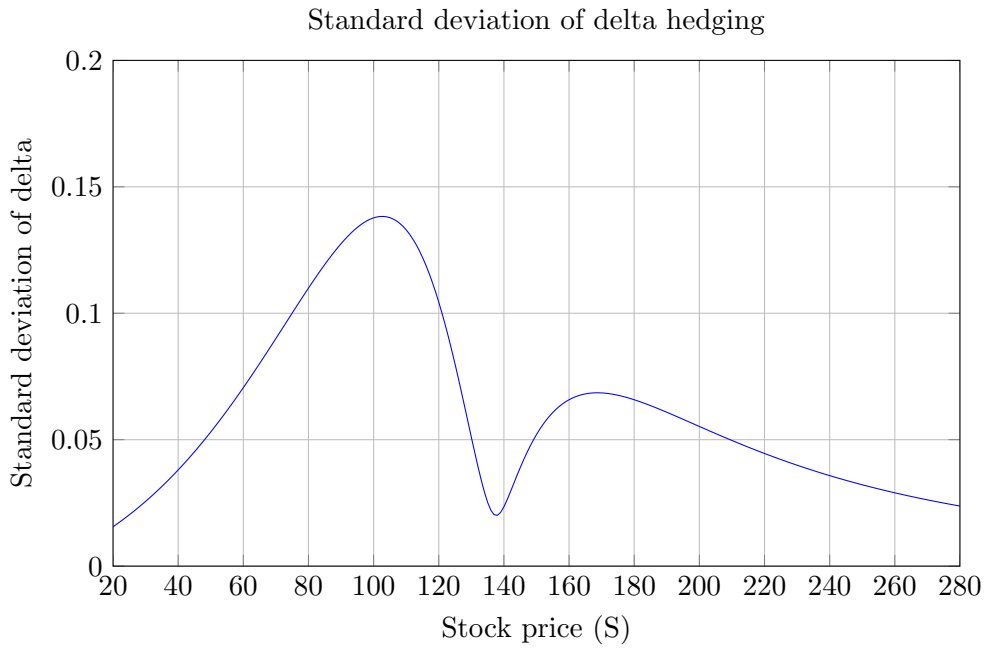
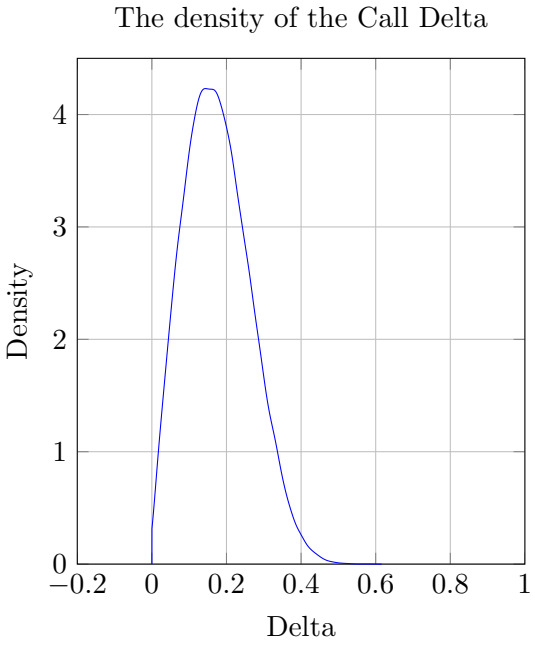
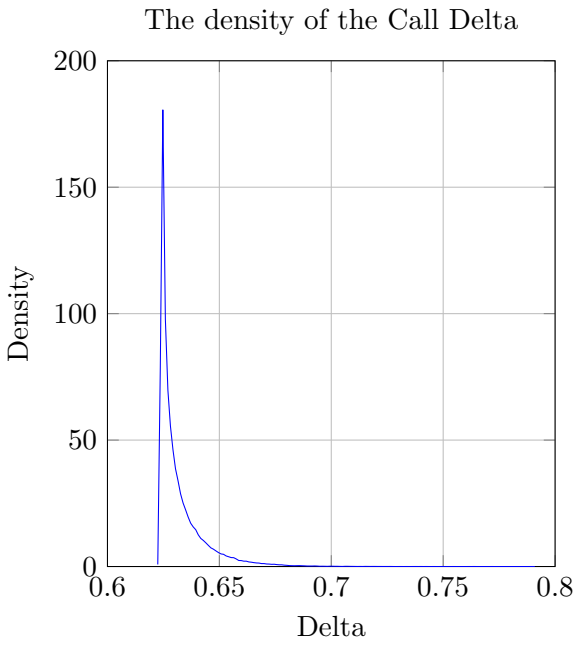


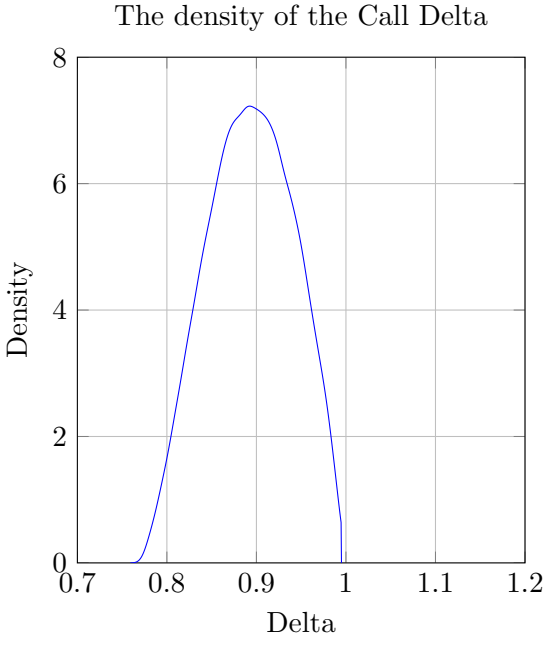
Figure 4.8: The standard deviation of the Call Delta



(a) $\Delta(100, 0; \sigma)$

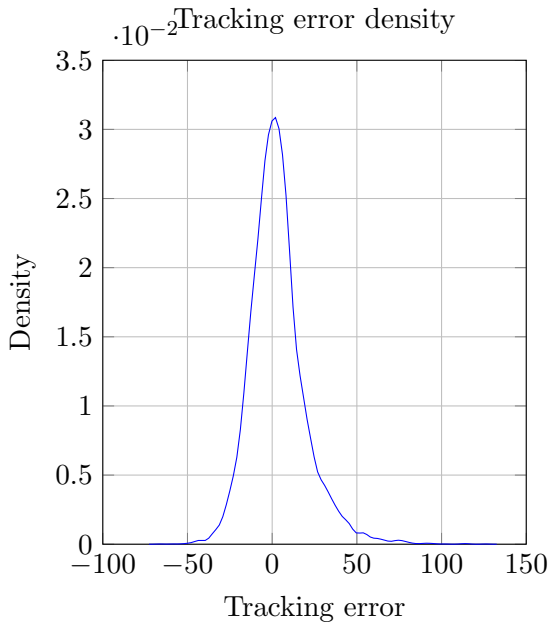


(b) $\Delta(140, 0; \sigma)$

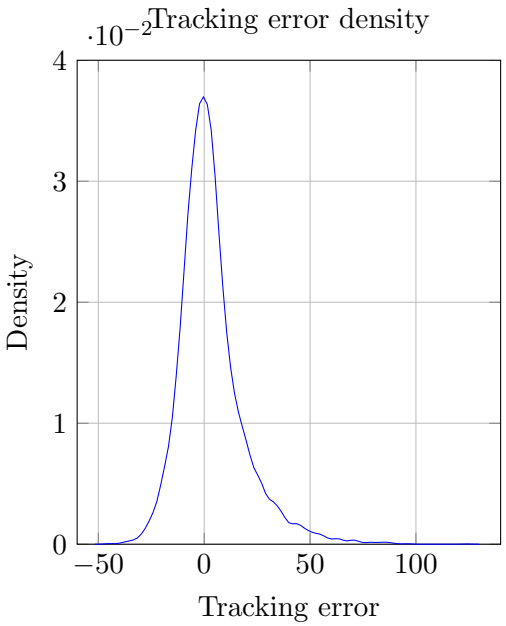


(c) $\Delta(180, 0; \sigma)$

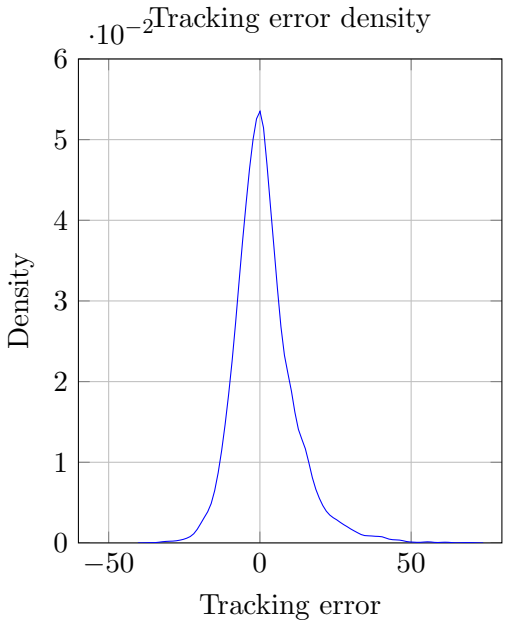
Figure 4.9: Call option Delta density



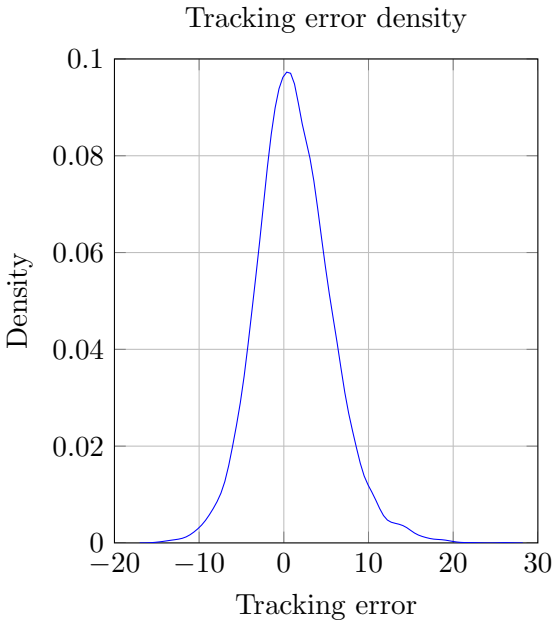
(a) $N = 10$



(b) $N = 20$



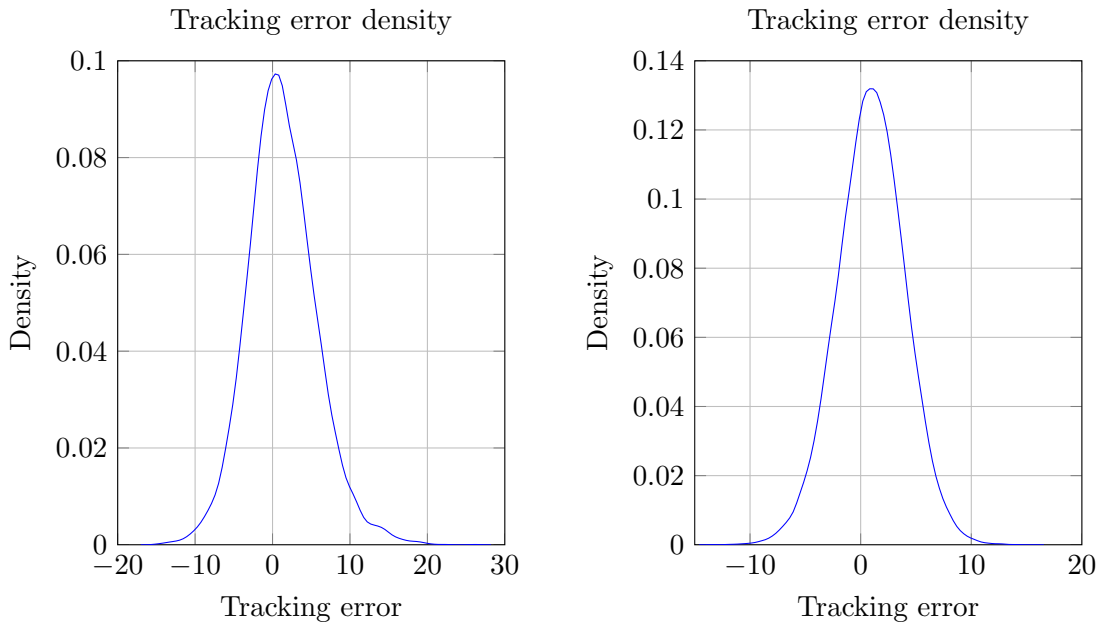
(c) $N = 40$



(d) $N = 100$

Figure 4.10: Tracking error density at maturity for different time steps

We note that the tracking error density looks like a normal one centered around zero. In addition to that, the more the hedging interval becomes smaller, the more the density of the tracking error tends to be more concentrated around zero.
We also observe that when the uncertainty in the volatility tends to be reduced,i.e. the standard deviation of the volatility decreases (in figure 4.11, the reduction ratio is $\frac{1}{10}$), the tracking error density for the same number of time steps becomes more centered around zero as shows figure 4.11.



(a) Initial uncertainty in the volatility

(b) Reduced uncertainty in the volatility

Figure 4.11: Effect of the uncertainty reduction on the tracking error density at maturity for $N = 100$

4.1.4 Cost analysis

The main advantage of Polynomial Chaos over the standard Monte Carlo method comes from the reduction of the computational cost. In fact, we could analyze the cost of each computation.
Let τ_1 denotes the computation time of the option solver and τ_2 the one of computing the PC expansion. On could write the cost of each method as follow:

$$\begin{aligned} C_{MC} &= N\tau_1 \\ C_{PC} &= J\tau_1 + N\tau_2 \end{aligned} \tag{4.1}$$

From equations in 4.1, one could deduce that:

$$\frac{C_{PC}}{C_{MC}} = \frac{J\tau_1 + N\tau_2}{N\tau_1} = \frac{J}{N} + \frac{\tau_2}{\tau_1} \tag{4.2}$$

As we can see, equation 4.2 has two terms: $\frac{J}{N}$ and $\frac{\tau_2}{\tau_1}$. In general, we have $J \ll N$. This first term is independent from the option solver. For the second term, in order for the PC method to be more cost efficient, the term τ_1 should be larger than τ_2. In our case, we have used the parameters $N = 100000$ and $J = 50$. For the times, we got $\tau_1 = 1s$ and $\tau_2 = 0.06$. Therefore, we have:

$$\frac{C_{PC}}{C_{MC}} = \frac{50}{100000} + \frac{0.06}{1} = 0.0605$$

In this way, the PC method is 16.5 times faster than the Crude Monte Carlo method.

The existence of an exact solution for the European Call has significantly reduced the benefit of the Polynomial Chaos method. When quantifying the uncertainty in other types of options which are more complicated, this ratio becomes greater. In the field of computational fluid dynamics where problems are as complex as in pricing exotic options, the speed-up ratio could reach 24000, for more details see [**?**].

4.2 Uncertainty quantification in basket options

We first analyzed in the previous section the effect of uncertainty in the volatility on the price of a European Call under the Black and Scholes model as well as the Delta hedging strategy. This section is a generalization of the 1D case; we are interested now in quantifying the effect of the uncertainty in the case of a basket option.

4.2.1 Data presentation

The data we will be using in our case are taking from the same source finance.yahoo.com. The derivative now is a Call basket option consisting of 5 assets. Table 4.3 presents these assets. In fact, all of them are parts from the main German stock index DAX.

Table 4.3: Basket option assets

Asset	Field	Ticker
Volkswagen Group	Automobile manufacturer	VOW.DE
Siemens	Engineering conglomerate	SIE.DE
Allianz	Insurance	ALV.DE
Bayer	Pharmaceutical Industry	BAYN.DE
Deutsche Bank	Bank	DBK.DE

The estimation of the parameters mainly the volatilities and the correlations between the assets were computed using the historical data for the year 2012. Ideally, we should have computed implied volatility not the historical one. However, due to the lack of access to the option prices and following a number of studies that were conducted in the basket

option pricing, we find that this choice is reasonable to a certain extent.
The results of the estimation yielded the following parameters:

Table 4.4: The data used for the basket Call option

Asset	Volatility	Correlation				
Volkswagen Group	0.2912	1	0.529	0.5085	0.4691	0.4533
Siemens	0.1915	0.529	1	0.6698	0.5973	0.6484
Allianz	0.2407	0.5085	0.6698	1	0.6868	0.7823
Bayer	0.2502	0.4691	.5973	0.6868	1	0.5626
Deutsche Bank	0.3978	0.4533	0.6484	0.7823	0.5626	1

In addition to that, we will be using the following variables for the simulations:

- The initial assets prices: $S(0) = [106.2 \quad 75.44 \quad 76.71 \quad 50.81 \quad 30.35]$
- The strike: $K = 70$.
- The expiry: $T = 1$ year.
- The risk neutral rate: $r = 0.05$.
- The weight of each asset: $\omega_i = 0.2$

In the following sections, our interest will be focused on determining the impact of the uncertainty present in the covariance matrix on the basket option pricing as well as hedging strategy. The uncertain parameter in the multidimensional case is no longer a scalar but rather a whole matrix.

4.2.2 Uncertainty quantification in the basket option pricing

The first step consists on comparing the approximation methods developed in the section 1 of chapter 3. The aim behind this comparison is to choose the most adequate approximation in order to be used for the rest of the computations.
Table 4.5 shows the results of the numerical simulations using the approximation methods.

Table 4.5: Call basket option price using approximation methods

Ju	Beisser	Reciprocal Gamma	Lognormal	Monte Carlo	Confidence Interval
6.3818	6.2819	6.62	6.7272	6.4023	$[6.3549, 6.3938]$

As we can notice from the table 4.5, the best approximation method is Ju's approximation. In the rest of this report, we will be using only this method in order to approximate the basket option price.

Using the Wishart distribution to represent the uncertainty in the covariance matrix and taking into consideration the variable reduction technique described in section 3.*b*) of chapter 3, we developed the following approaches to quantify the impact of the uncertainty on the basket option price using:

1. Monte Carlo simulation with one variance reduction technique: control variates.
2. Monte Carlo simulation with two variance reduction techniques: control variates and antithetic variates.
3. Ju's approximation combined with Monte Carlo method.
4. Ju's approximation coupled with Sparse Grid technique.

The numerical tests of these approaches are summarized in table 4.6. For the Monte Carlo method, the number of simulations is $M = 100000$.

Table 4.6: Call basket option price

Approach	Price	Standard Error	Confidence Interval
1.	6.3861	0.0119	$[6.3665, 6.4056]$
2.	6.3789	0.0095	$[6.3603, 6.3975]$
3.	6.3764	0.0078	$[6.3611, 6.3916]$
4.	6.3804		

Here the standard error is nothing but the square root of the ratio of the standard deviation of the Monte Carlo simulations to the number of simulations.

Our use of sparse grid technique, as it is been justified in section 2 of chapter 3, is due essentially to its relatively fast convergence rate comparing it with the Monte Carlo convergence rate as we can see in figure 4.12.

Figure 4.13 presents the basket option price density. As you can notice, the density has almost a Gaussian shape. Therefore, using a Taylor expansion to the second order can be used in this case to compute the price statistics and density since beyond the second order, the error is negligible. Figure 4.14 compares the results using the moments method based on Taylor expansion and the Monte Carlo method.

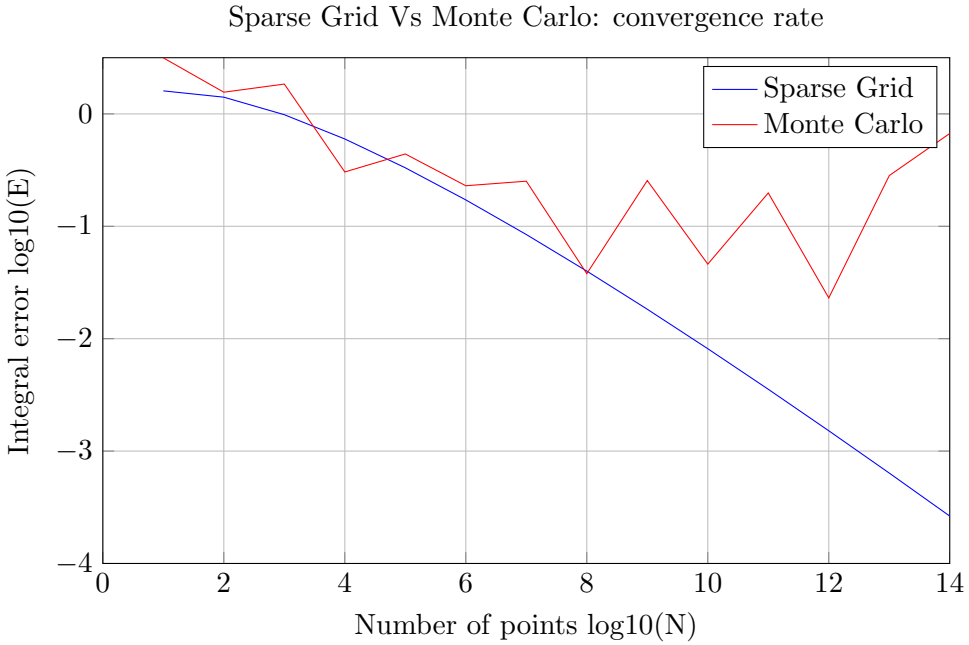


Figure 4.12: Convergence rate

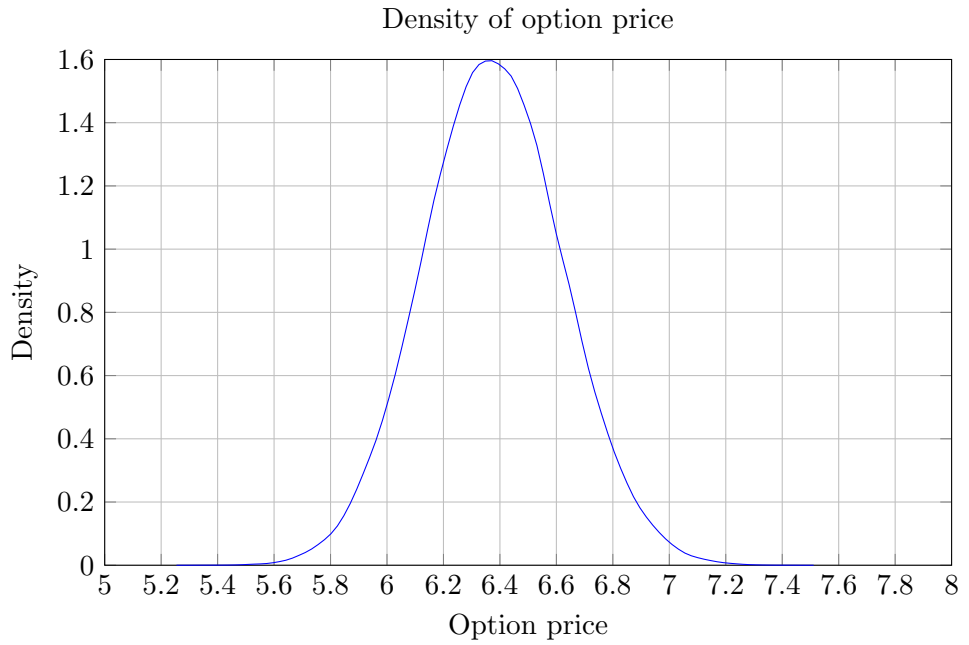


Figure 4.13: Basket option price via Monte Carlo

As we can see clearly in the figure 4.14 the moments method yields an interesting result and the density curve plotted using this method almost match the one with Monte Carlo.

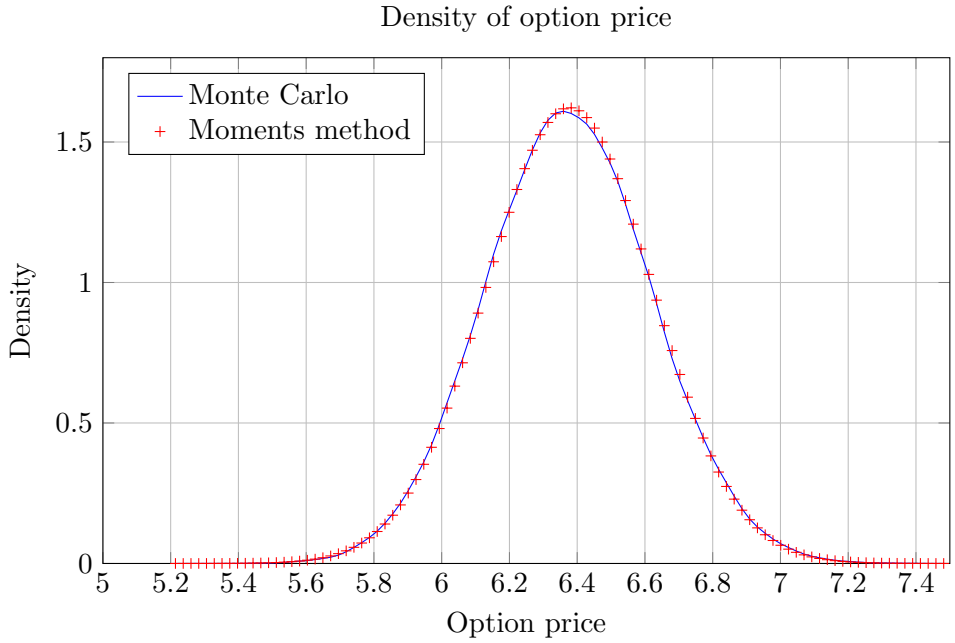


Figure 4.14: Comparison between moments method and Monte Carlo: basket option price

In addition to that, the huge difference in the computational time is in favor of the moments method as you can see in table 4.7. However, we should note that our use for the Taylor Young expansion in this case is justified because of the Gaussian shape of the price density.

Table 4.7: Computational time comparison

	Monte Carlo	Moments method
CPU time (s)	150	5

4.2.3 Uncertainty quantification in the basket option hedging

As in the 1D case, our hedging strategy is based essentially on the Delta parameter. Therefore, we will try at the first place to see the influence of the uncertainty in the covariance matrix estimation on the Deltas of the basket assets as a first step and then on the Delta hedging strategy.

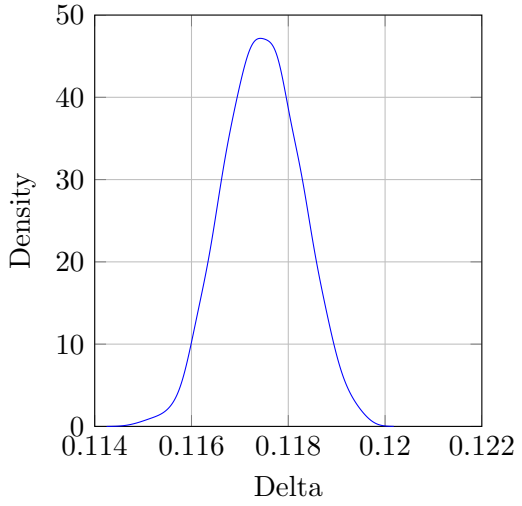
Our focus will be mainly on plotting the density functions of the Deltas of the basket option. Figure 4.15 shows that the Deltas densities have almost a Gaussian shape with some minor deformations at the tails. These densities are computed at the initial time $t = 0$ and using a finite difference method based on the use of Ju's approximation formula for computing the basket price.

Table 4.8 presents the value, the standard error and the confidence interval relative to each Delta of the basket option's assets.

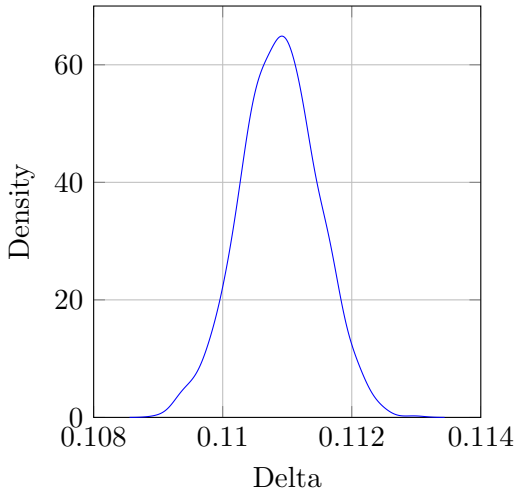
Table 4.8: Deltas of the basket option

Asset n^o	Delta value	Standard Error	Confidence Interval
1	0.1175	$0.0780\ 10^{-2}$	$[0.1159, 0.1190]$
2	0.1108	$0.0611\ 10^{-2}$	$[0.1096, 0.1119]$
3	0.1149	$0.0597\ 10^{-2}$	$[0.1137, 0.1160]$
4	0.1137	$0.0806\ 10^{-2}$	$[0.1121, 0.1152]$
5	0.1232	$0.1189\ 10^{-2}$	$[0.1208, 0.1255]$

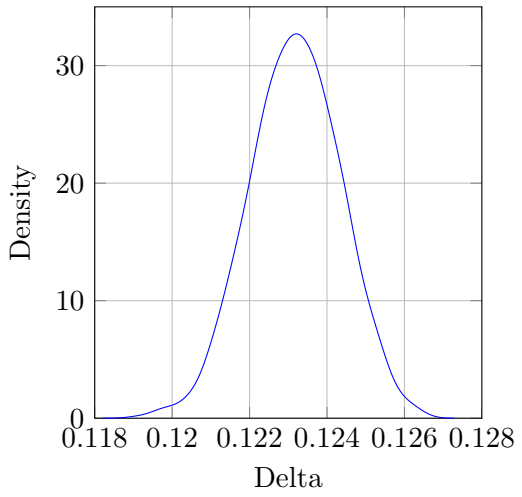
Once we were able to compute the effect of the uncertainty in the volatility on the hedging parameter Delta, we will be interested in measuring its impact on the Delta hedging strategy. Therefore, we plotted the tracking error density at the maturity as you can see in the figure 4.16 using $N = 200$ as hedging intervals. As we can see, the tracking error density is almost normal. However, unlike the 1D case, we notice that the curve is not really centered around zero but rather around 5.



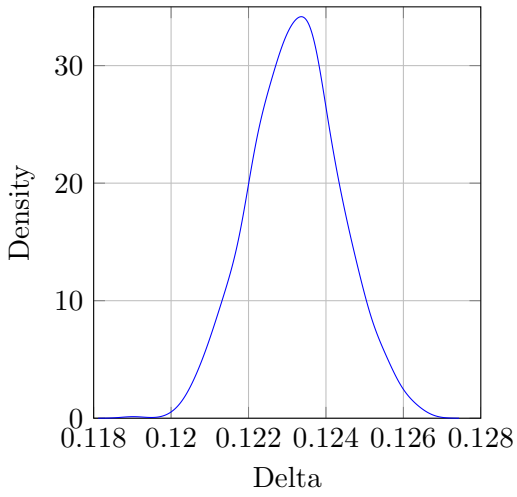
(a) Δ_1



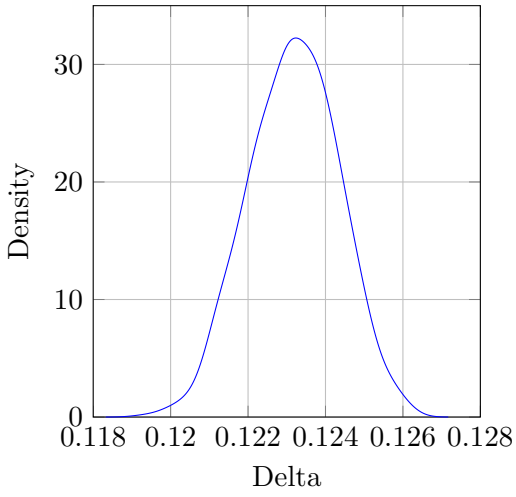
(b) Δ_2



(c) Δ_3



(d) Δ_4



(e) Δ_5

Figure 4.15: Basket option Deltas density functions

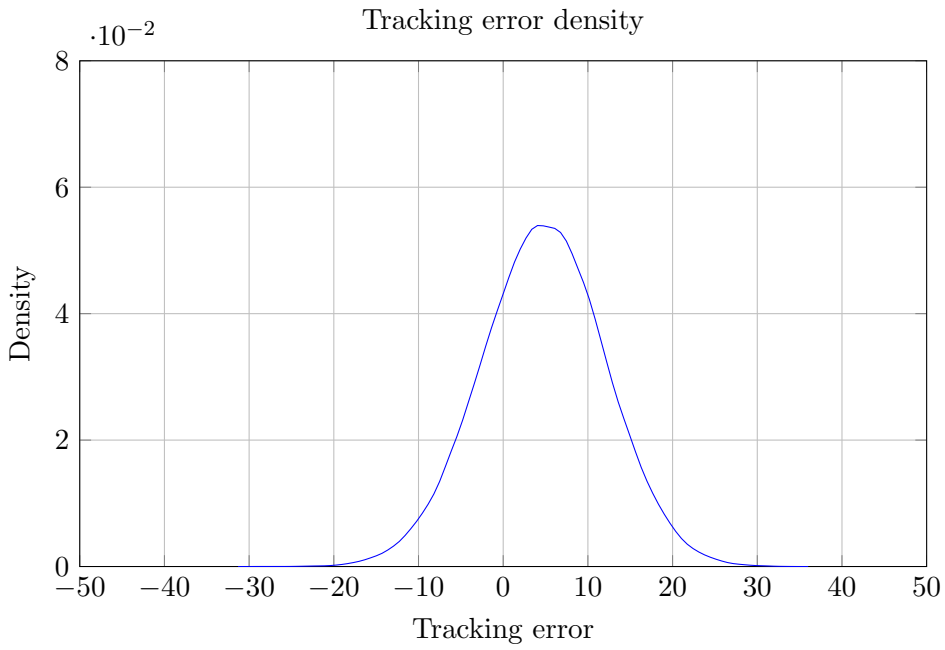


Figure 4.16: Tracking error density at maturity

Conclusion

This report aims to quantify the effect of uncertainty in the volatility on the European options pricing and Delta hedging strategy. Two main methods were developed to achieve this goal: Polynomial Chaos coupled with Monte Carlo and sparse grid. As an application, we worked with the European Call for the one dimension case, the basket Call for the multidimensional case.

The contribution of the present report in the financial field is essentially in the multidimensional case since we were able to represent the uncertainty in the covariance matrix via Wishart distribution and propagated to the basket option and hedging strategy using sparse grid method. Using a dimension reduction technique enabled us to reduce significantly the computational cost of our simulations.

In the light of this thesis, the conducted study can be guidance for future research. In fact, many possible extensions can be done. We can extend our work for other types of exotic options. Another possible extension is to model the volatility as a stochastic process and try to quantify the effect of uncertainty in the model parameters that are generally estimated and therefore presents a certain uncertainty. In addition to that, we could have improved the hedging strategy by using a Gamma-Delta rather than a Delta hedging strategy. There is also the possibility of modeling the effect not of the volatility but rather of the interest rate on the fixed income products.

Appendices

Appendix I: Taylor Ju' approximation

The following appendix main aim is to provide the expression of the terms $(d_i)_{i=1,\dots,4}$ used in the Taylor Ju's approximation of the Call basket option price.

We begin by determining the expansion of $f(z)$ around $z = 0$ up to z^6. For that, let's recall the expression of $f(z)$:

$$f(z) = \frac{E\left[e^{i\phi X(z)}\right]}{E\left[e^{i\phi Y(z)}\right]} = E\left[e^{i\phi X(z)}\right] e^{-i\phi m(z)+\phi^2\frac{v(z)}{2}}$$

First, we expand $e^{-i\phi m(z)+\phi^2\frac{v(z)}{2}}$. Since $v'(z) = -2m'(z)$, we have:

$$\begin{aligned} e^{-i\phi m(z)+\phi^2\frac{v(z)}{2}} &\approx e^{-i\phi m(0)+\phi^2\frac{v(0)}{2}-(i\phi+\phi^2)m'(0)z^2-(i\phi+\phi^2)m''(0)\frac{z^4}{2}-(i\phi+\phi^2)m^{(3)}(0)\frac{z^6}{6}} \\ &\approx e^{-i\phi m(0)+\phi^2\frac{v(0)}{2}}[1-(i\phi+\phi^2)a_1+\frac{1}{2}((i\phi+\phi^2)^2a_1^2-(i\phi+\phi^2)a_2) \\ &+\frac{1}{6}(3(i\phi+\phi^2)^2a_1a_2-(i\phi+\phi^2)a_3-(i\phi+\phi^2)^3a_1^3)] \end{aligned} \tag{4.3}$$

where:

$$\begin{aligned} a_1 &= z^2m'(0) \\ a_2 &= z^4m''(0) \\ a_3 &= z^3m^{(3)}(0) \end{aligned}$$

Now, we will try to expand the second term $g(z) = E\left[e^{i\phi X(z)}\right]$. In order to do that, we have to calculate the derivative of $g(z)$ with respect to z. We start by differentiating $g(z)$ twice:

$$g''(z) = E\left[e^{i\phi X(z)}\left(-(i\phi+\phi^2)(X'(z))^2+i\phi\frac{A''(z)}{A(z)}\right)\right] \tag{4.4}$$

Noting that $E\left[\frac{A''(0)}{A(0)}\right] = 0$ and $z^2E[(X'(z))^2] = \frac{z^2}{A^2(0)}E[(A'(0))^2] = -2a_1(z)$, the Equation 4.4 yields:

$$\frac{z^2}{2}g''(0) = e^{i\phi X(0)}(i\phi+\phi^2)a_1(z)$$

Differentiating $g(z)$ four times yields:

$$\begin{aligned} g^{(4)}(z) &= E\left[e^{i\phi X(z)}\left(-(i\phi-3)(i\phi-2)(i\phi+\phi^2)(X'(z))^4 - 6(i\phi-2)(i\phi+\phi^2)\frac{A''(z)(A'(z))^2}{A^3(z)}\right.\right. \\ &\quad \left.\left. -3(i\phi+\phi^2)\left(\frac{A''(z)}{A(z)}\right)^2 - 4(i\phi+\phi^2)\frac{A'(z)A^{(3)}(z)}{A^2(z)} + i\phi\frac{A^{(4)}}{A(z)}\right)\right] \end{aligned} \tag{4.5}$$

The fact that $X'(0)$ is normally distributed with mean zero yields:

$$z^4E[(X'(0)^4] = 3z^4(E[(X'(0)^2])^2 = 12a_1^2(z)$$

Noting also that $E\left[\frac{A'(0)A^{(3)}(0)}{A^2(0)}\right] = 0$ and $E\left[\frac{A^{(4)}(0)}{A(0)}\right] = 0$ and using Equation 4.5 for $z = 0$ gives:

$$\frac{z^4}{24}g^{(4)}(0) = e^{i\phi X(0)}(i\phi+\phi^2)(-(i\phi-3)(i\phi-2)\frac{a_1^2(z)}{2} - (i\phi-2)b_1(z) - b_2(z))$$

where:

$$b_1(z) = \frac{z^4}{4A^3(0)}E[(A'(0))^2A''(0)]$$
$$b_2(z) = \frac{z^4}{8A^2(0)}E[(A''(0))^2]$$

The last step is to calculate the sixth derivative of $g(z)$:

$$\begin{aligned} g^{(6)}(z) &= E[e^{i\phi X(z)}(-(i\phi-5)(i\phi-4)(i\phi-3)(i\phi-2)(i\phi+\phi^2)(X'(z))^6 \\ &\quad -15(i\phi-4)(i\phi-3)(i\phi-2)(i\phi+\phi^2)\frac{A''(z)(A'(z))^4}{A^5(z)} \\ &\quad -(i\phi-3)(i\phi-2)(i\phi+\phi^2)((45\frac{A''(z)A'(z)}{A^2(z)})^2 + 20\frac{(A'(z))^3A^{(3)}(z)}{A^4(z)} \\ &\quad -(i\phi-2)(i\phi+\phi^2)(15(\frac{A'(z)}{A(z)})^2\frac{A^{(4)}(z)}{A(z)} + 60\frac{A'(z)A''(z)A^{(3)}(z)}{A^{(3)}(z)} + 15\frac{A''(z)}{A(z)})^3) \\ &\quad -(i\phi+\phi^2)(6\frac{A'(z)A^{(5)}(z)}{A^2(z)} + 10(\frac{A^{(3)}(z)}{A(z)})^2 + 15\frac{A''(z)A^4(z)}{A^2(z)}) + i\phi\frac{A^{(6)}(z)}{A(z)})] \end{aligned}$$

Taking into consideration the following equations:

$$
\begin{aligned}
&E\left[\frac{(A'(0))^2 A^{(4)}(0)}{A^3(0)}\right] = 0 \\
&E\left[\frac{A'(0)A^{(5)}(0)}{A^2(0)}\right] = 0 \\
&E\left[\frac{A''(0)A^{(4)}(0)}{A^2(0)}\right] = 0 \\
&E\left[\frac{A^{(6)}(0)}{A(0)}\right] = 0 \\
&z^6 E\left[(X'(0))^6\right] = -120 a_1^3(z)
\end{aligned}
$$

One can deduce the following expression:

$$
\begin{aligned}
\frac{z^6}{720} g^{(6)}(z) \;=\; & E[e^{i\phi X(0)}(i\phi+\phi^2)((i\phi-5)(i\phi-4)(i\phi-3)(i\phi-2)\frac{a_1^3}{6} \\
& -(i\phi-4)(i\phi-3)(i\phi-2)c_1(z) - (i\phi-3)(i\phi-2)c_2(z) \\
& -(i\phi-2)c_3(z) - c_4(z)]
\end{aligned}
$$

where:

$$
\begin{aligned}
c_1(z) &= \frac{z^6}{48A^5(0)} E[(A'(0))^4 A''(0)] \\
c_2(z) &= \frac{z^6}{144A^4(0)} (9E[(A'(0))^2 (A''(0))^2] + 4E[(A'(0))^3 (A''(0))^3]) \\
c_3(z) &= \frac{z^6}{48A^3(0)} (4E[(A'(0))^2 A''(0) A^{(3)}(0)] + E[(A''(0))^3]) \\
c_4(z) &= \frac{z^6}{72A^2(0)} E[(A^{(3)}(0))^2]
\end{aligned}
$$

Finally, we have:

$$
g(z) \approx g(0) + \frac{z^2}{2} g''(0) + \frac{z^4}{24} g^{(4)}(0) + \frac{z^6}{720} g^{(6)}(0) \tag{4.6}
$$

Multiplying the expansion of 4.3 and the one of 4.6, we could give the expansion of $f(z)$ around $z = 0$ up to the order of z^6:

$$
f(z) \approx 1 - i\phi d_1(z) - \phi^2 d_2(z) + i\phi^3 d_3(z) + \phi^4 d_4(z)
$$

where:

$$
\begin{aligned}
d_1(z) &= \frac{1}{2}(6a_1^2(z) + a_2(z) - 4b_1(z) + 2b_2(z)) - \frac{1}{6}(120a_1^3(z) - a_3(z) + 6(24c_1(z) - 6c_2(z) \\
&\quad +2c_3(z) - c_4(z))) \\
d_2(z) &= \frac{1}{2}(10a_1^2(z) + a_2(z) - 6b_1(z) + 2b_2(z)) - (128\frac{a_1^3(z)}{3} - \frac{a_3(z)}{6} + 2a_1(z)b_1(z) - a_1(z) \\
&\quad b_2(z) + 50c_1(z) - 11c_2(z) + 3c_3(z) - c_4(z))) \\
d_3(z) &= 2a_1^2(z) - b_1(z) - \frac{1}{3}(88a_1^3(z) + 3a_1(z)(5b_1(z) - 2b_2(z)) + 3(35c_1(z) - 6c_2(z) + c_3(z))) \\
d_4(z) &= -\frac{20}{3}a_1^3(z) + a_1(z)(-4b_1(z) + b_2(z)) - 10c_1(z) + c_2(z)
\end{aligned}
$$

Appendix II: Smoylak algorithm: a numerical example

In this appendix, we present a numerical example of the Smoylak algorithm in the bidimensional space. The illustration uses Gauss-Hermite quadrature up to the level $l = 2$. We recall the definition of the classical 1D formula:

$$\int_{\Omega} f(x)\rho(x) \approx U_{1,l}f := \sum_{i=1}^{m_l} \omega_{i,l} f(x_{i,l}) \tag{4.7}$$

The following table gives the nodes and weights of Gauss Hermite quadrature for the first two levels.

Table 4.9: Nodes and weights of Gauss Hermite quadrature

l	n_l	$x_{i,l}$	$\omega_{i,l}$
1	1	0	$\sqrt{\pi}$
2	3	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{\pi}}{6}$
		0	$\frac{2}{3}\sqrt{\pi}$
		$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{\pi}}{6}$

Case $l = 1$:
Using the definition 4.7, we can write:

$$U_{2,1}f = \sum_{|k|_1 \leqslant 2} \Delta_{2,k} f \quad \text{where} \quad \Delta_{2,k} f = (\Delta_{1,k_1} \otimes \Delta_{1,k_2}) f \quad k \in \mathbb{N}^2 \tag{4.8}$$

The admissible set of k is $S = \{(0,0)(0,1)(1,0)(0,2)(2,0)(1,1)\}$. Since $\Delta_0 f = 0$, we keep only the pairs having non null elements. In this way, the set S is reduced to $S' = \{(1,1)\}$. The equation 4.8 becomes:

$$U_{2,1}f = \Delta_{2,(1,1)} f = (\Delta_{1,1} \otimes \Delta_{1,1}) f \tag{4.9}$$

Since $\Delta_{1,1}f = U_{1,1}f - U_{1,0}f = U_{1,1}f = \omega_{1,1} f(x_{1,1})$, the equation 4.9 can be written as follow:

$$U_{2,1}f = \omega_{1,1} f(x_{1,1}) \otimes \omega_{1,1} f(x_{1,1}) = \omega_{1,1}\omega_{1,1} f(x_{1,1}) \otimes f(x_{1,1})$$

Using the table 4.9, we obtain:

$$U_{2,1}f = \pi f(0,0)$$

Case $l = 2$:
In this case, the equation 4.8 becomes:

$$U_{2,2}f = \sum_{|k|_1 \leqslant 3} \Delta_{2,k} f \tag{4.10}$$

The new set of admissible k is $S = \{(0,0)(0,1)(1,0)(0,2)(2,0)(1,1)(1,2)(2,1)(0,3)(3,0)\}$. For the sake of the same argument, i.e. excluding elements having at least one component is zero, the new set is $S' = \{(1,1)(1,2)(2,1)\}$ and the equation 4.10 becomes:

$$U_{2,2}f = \Delta_{2,(1,1)}f + \Delta_{2,(1,2)}f + \Delta_{2,(2,1)}f \tag{4.11}$$

Since, we already computed the term $\Delta_{2,(1,1)}f$, we will calculate only the terms $\Delta_{2,(1,2)}f$ and $\Delta_{2,(2,1)}f$. To do that, we recall that we have:

$$\begin{aligned} \Delta_{2,(1,2)}f &= (\Delta_{1,1} \otimes \Delta_{1,2})f \\ \Delta_{2,(2,1)}f &= (\Delta_{1,2} \otimes \Delta_{1,1})f \end{aligned} \tag{4.12}$$

The term $\Delta_{1,1}$ has been computed in the first case, so we will calculate only the term $\Delta_{1,2}$:

$$\begin{aligned} \Delta_{1,2} &= (U_{1,2} - U_{1,1})f \\ &= \sum_{i=1}^{3} \omega_{i,2} f(x_{i,2}) - \omega_{1,1} f(x_{1,1}) \end{aligned}$$

Equations 4.12 can be written now as follow:

$$\begin{aligned} \Delta_{2,(1,2)}f &= \omega_{1,1}f(x_{1,1}) \otimes (\omega_{1,2}f(x_{1,2}) + \omega_{2,2}f(x_{2,2}) + \omega_{3,2}f(x_{3,2}) - \omega_{1,1}f(x_{1,1}) \\ \Delta_{2,(2,1)}f &= (\omega_{1,2}f(x_{1,2}) + \omega_{2,2}f(x_{2,2}) + \omega_{3,2}f(x_{3,2}) - \omega_{1,1}f(x_{1,1}) \otimes \omega_{1,1}f(x_{1,1}) \end{aligned}$$

The final approximation of the integral can be written now:

$$\begin{aligned} U_{2,2}f \quad = \quad & -\omega_{1,1}\omega_{1,1}f(x_{1,1}, x_{1,1}) + \omega_{1,1}\omega_{1,2}f(x_{1,1}, x_{1,2}) + \omega_{1,1}\omega_{2,2}f(x_{1,1}, x_{2,2}) + \omega_{1,1}\omega_{3,2}f(x_{1,1}, x_{3,2}) \\ & +\omega_{1,2}\omega_{1,1}f(x_{1,2}, x_{1,1}) + \omega_{2,2}\omega_{1,1}f(x_{2,2}, x_{1,1}) + \omega_{3,2}\omega_{1,1}f(x_{3,2}, x_{1,1}) \end{aligned}$$

If we use Gauss Hermite quadrature, the last equation becomes:

$$U_{2,2}f = -\frac{\pi}{3}f(0,0) + \frac{\pi}{6}(f(0, \frac{\sqrt{6}}{2}) + f(0, -\frac{\sqrt{6}}{2}) + f(\frac{\sqrt{6}}{2}, 0) + f(-\frac{\sqrt{6}}{2}, 0))$$

Appendix III: Numerical tests using sparse grid method

In this appendix, we will performs some numerical tests for integral computation using the sparse grid method. To begin with, let's define the integrals that we aim to calculate. Definition of the integrals:

$$I_1 = \int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \cos(x(y+z))dxdydz$$

$$I_2 = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \cos(xyz)dxdydz$$

$$I_3 = \int_1^2\int_1^2\int_1^2\int_1^2 (x - \frac{y}{z} + w^2)dxdydzdw$$

$$I_4 = \int_1^2\int_1^2\int_1^2\int_1^2 (x - \frac{y}{z} + w^2)exp(a)dxdydzdwda$$

$$I_5 = \int_1^3\int_1^3\int_1^3\int_1^3\int_1^3 \frac{\sqrt{\frac{x}{1+y}}\exp(a+\frac{z}{w})}{t}dxdydzdwdadt$$

$$I_d = \int_{R^d} \exp(-\sum_{i=1}^{d} x_i^2)dx$$

The exact values of the previous integrals are:

$$I_1 = \frac{-2\pi^2\left(\mathrm{Si}\left(\pi^2\right) - \mathrm{Si}\left(2\pi^2\right)\right) + 1 - 2\cos\left(\pi^2\right) + \cos\left(2\pi^2\right)}{\pi}$$

$$I_2 = 8\pi^3\,{}_2F_3\left(\frac{1}{2},\frac{1}{2};\frac{3}{2},\frac{3}{2},\frac{3}{2};-\frac{\pi^6}{4}\right)$$

$$I_3 = \frac{1}{6}(23 - 9\log(2))$$

$$I_4 = -\frac{1}{6}(e-1)e(\log(512)-23)$$

$$I_5 = -\frac{2}{3}(-2+\sqrt{2})(-1+3\sqrt{3})e(-1+e^2)(-4(3\sqrt[3]{e}-5e+e^3)+Ei(1/3)-10Ei(1)+9Ei(3))\log(3)$$

$$I_d = \sqrt{\pi}^d$$

where:

- Si(x) is the sine integral.
- Ei(x) is the exponential integral.
- ${}_pF_q$ is the generalized hypergeometric function.

Integral	Approximated value	Sparse grid	Level
I_1	0.2913	0.2913	14
I_2	50.4128	50.4128	37
I_3	2.79361	2.79361	5
I_4	13.0483	13.0483	6
I_5	862.0514	862.0514	11
$I_{d=2}$	3.1416	3.1416	3
$I_{d=5}$	17.4934	17.4934	6
$I_{d=8}$	97.4091	97.4091	9
$I_{d=10}$	306.0197	306.0197	11

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