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## Graduation Project Report

# Numerical Methods For Uncertainty Quantification In Option Pricing

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To my family for their love and support
To my friends for their kindness and advice
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**Abstract**

Model or/and parametric uncertainty, in the context of derivative pricing, results in mispricing of contingent claims due to uncertainty on the choice of the pricing models or/and the values of parameters within these models. In this thesis, we introduce a quantitative framework for investigating the impact of parametric uncertainty on option pricing under Black-Scholes framework. We start with one dimensional case by quantifying the impact of the volatility parameter on the price of European put. From historical data, we fit the sample density of the volatility by a parametric distribution. We then use Monte Carlo Sampling (MCS) and construct Polynomial Chaos (PC) approximation to compute statistical information (i.e, mean quantities, $\alpha$ level confidence bounds ($\alpha$ bounds), and sample densities) of option's price. We show that both methods give the same results but PC approximation method performs significantly better. In the second part of our project, we extend the work to multidimensional case by quantifying the impact of the covariance matrix on the price of European basket option. After modeling the randomness in the covariance matrix with a Wishart distribution, we developed three ways to compute mean quantities and sample densities of basket put price: a nested MC simulation and two methods, based on Monte Carlo (MC) and Sparse Grid Quadrature (SGQ) techniques, that use an approximation of basket option price. We show that the three methods give the same results, those using approximation of basket option price are more efficient and the performance of SGQ decreases for high dimensional problems.

**Keywords:** parametric uncertainty, option pricing, Monte Carlo Sampling, Polynomial Chaos, Sparse Grid Quadrature.

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# Chapter 1

# Introduction

## 1.1 Motivation

Computational finance has emerged in the last years as a discipline able to simulate the behavior of complex financial systems described by sophisticated mathematical models. All this is made possible by both the exponential growth of computer power and new mathematical models that we have witnessed in the last decades. However, the robustness of any financial model depends in part on the accuracy and reliability of its output. Yet, because all models are imperfect abstractions of reality, and because precise input data are rarely if ever available, all output values are subject to imprecision. Model or/and parametric uncertainty results in mispricing of contingent claims due to uncertainty on the choice of the pricing models or/and the values of parameters within these models.

In fact, mathematical models in financial industry are either parametric or nonparametric. Parametric models are the dominating approach, as these are easier to analyze and to fit to data. A limitation and simultaneously the strength of these models is their limited flexibility, resulting in low variance and some bias; however, nonparametric models are flexible and less biased but often poor predictors since they need more data to produce a stable fit.

Popular parametric option pricing models include the Black-Scholes model [1], the Merton jump diffusion [2] model, the Heston stochastic volatility model [3] etc. They are calibrated to data by estimating the parameters either by minimizing some loss function [4] or through nonlinear Kalman filters [5]. It is easily argued that inaccurate calibration methods can cause problems. The complexity of real-world calibration is due to the several subjective choices needed to be made. In fact, two things are needed, an estimator and a set of data. It is highly unlikely, that all investors are using identical estimates as they take different estimators and data sets into consideration, thereby causing the existence of uncertainty in

models parameters.

In this thesis, we develop a quantitative framework for studying the impact of parameters' uncertainty on option pricing. We focus on both European single asset and basket options under Black-Scholes framework.

## 1.2 Project Framework

### 1.2.1 Objectives

The aim of this project is to investigate the effect of parameters uncertainty on the Black-Scholes price of both European single asset and basket options.

In the case of single asset option, we assume that we have uncertainty only in the underlying asset's volatility and apply probabilistic uncertainty methods which are: Monte Carlo (MC) and Polynomial Chaos (PC) approximation to the Black-Scholes model. On one hand, our goal is to compare the statistical information of the option price (i.e, the mean quantities, $\alpha$ bounds, and the sample density) obtained by both of these methods. On the other hand, we want to compare the efficiency of the crude MC versus the surrogate PC/MC implementation.

In the case of multi-asset option, we face two challenges:

- As there is no close formula of arithmetic basket option's price under Black-Scholes framework, the first challenge is finding the best method to approximate it. Therefore, we analyze the performance of different approximations in order to select the best one.

- The second challenge is quantifying the effect of uncertainty in the covariance matrix of assets' returns on the basket option's price via computing its mean and sample density. To overcome this challenge, we develop three methods based on MC and Sparse Grid Quadrature (SGQ).

### 1.2.2 Related Work

UQ was first introduced into option pricing by the use of deterministic methods which are mainly sensitivity derivatives and Worst case analysis.

Sensitivity derivatives have gained the highest popularity in the derivatives community. The so-called Greeks of an option are nothing but various partial derivatives of the option price

with respect to a parameter or variable of interest. Since differentiation is a local operation, sensitivity derivatives give the localized dependence of a model on a parameter. However, they provide little to no information about the overall uncertainty of the model's output in relation to a particular parameter's uncertainty.

Giving that sensitivity analysis makes no prediction concerning the model's output value, The so-called Worst Case analysis was introduced to provide more information. This method was introduced into option pricing in the form of the Uncertain Volatility model by Avellaneda et al. [6] and Lyons [7] (both in 1995).

Worst case analysis assumes that the input parameter lies in some interval, and then gives corresponding bounds on the model's output. The basic idea of this approach: Given a parameter input, say $\sigma$, form an interval in which $\sigma$ is assumed to lie, i.e. $\sigma_{min} < \sigma < \sigma_{max}$, where $\sigma_{min}$ and $\sigma_{max}$ represent minimal and maximal values $\sigma$ can take. The option price is then both maximized and minimized subject to this constraint to obtain the prices, namely

$$V_{min}(S,t) = \min_{\sigma_{min} \leq \sigma \leq \sigma_{max}} (V(S,t;\sigma)) \tag{1.1}$$

$$V_{max}(S,t) = \max_{\sigma_{min} \leq \sigma \leq \sigma_{max}} (V(S,t;\sigma)), \tag{1.2}$$

where S is the underling's current price, $\sigma$ is the volatility, and $V(S,t;\sigma)$ denotes the option price as a function of S and t, given $\sigma$. The values of $V_{min}$ and $V_{max}$ give the worst case bounds for the option price, given that $\sigma$ lies in $[\sigma_{min}, \sigma_{max}]$.

The Uncertain Volatility model requires solving for the two values $V_{min}(S,t)$ and $V_{max}(S,t)$ defined by Equations (1.1) and (1.2) respectively. The bounds $V_{min}$ and $V_{max}$ are found in [8] by solving a nonlinear version of the Black-Scholes PDE.

One major critic of deterministic methods is that they do not ask how the option prices change, given the fact that the parameters can be distributed according to some probability density function. This issue was solved by developing probabilistic uncertainty methods.

Probabilistic Uncertainty analysis was introduced to the option pricing problem by Pulch and van Emmerich [8]. The authors solved for the mean and standard deviation of the price of European and Asian options using both MC and PC methods assuming a uniform distribution for the parametric uncertainty. However, they did not consider other distributions that best fit uncertain parameters or consider a higher dimensional case such as basket options.

### 1.2.3 Main Contributions Of This Project

The main contributions of this project to uncertainty quantification (UQ) and option pricing are the following:

- Developing a one dimensional PC approximation corresponding to the parametric distribution that best fits the historical volatility data sample.
- Comparing the efficiency of MC versus surrogate PC/MC method for one dimensional case.
- Investigating the effect of uncertainty in the covariance matrix of assets' returns on the price of European Basket put option using MC and SGQ techniques.

## 1.3 Outline

This thesis is organized as follows. In *Chapter 2*, we recall the basics of pricing theory, develop the UQ framework , and illustrate the construction of SGQ with a brief literature review.

Then, *Chapter 3* will be dedicated to investigating the effect of volatility's uncertainty on European single asset put option's price. We develop a PC approximation method to compute the mean quantities, $\alpha$ bounds and the sample density of option price. We conclude this chapter with comparing the performance of crude MC and surrogate PC/MC methods.

In *Chapter 4*, the work will be extended to multidimensional cases where we investigate the impact of uncertainty in the covariance matrix of assets' returns on the basket option's price. After parameterizing the randomness in the covariance matrix and selecting the best approximation for basket option price, we develop three approaches based on MC and SGQ to compute the mean and sample density of basket put's price.

Concluding remarks are finally drawn at the end of this thesis. We reiterate the main results of the previous chapters and illustrate an outlook on possible extensions of our work.

# Chapter 2

# Basic Notions and Literature Review

## Introduction

Before getting into the details of our project, we should introduce the basic concepts related to pricing theory, UQ and the methods that we are going to implement in the course of our work. In the first section, we are going to develop the basic framework used to price European single asset and basket options and we present the main methods developed in the literature for pricing. The second chapter will be dedicated to UQ framework. We focus on probabilistic methods that we are going to use in our project which are precisely MC and PC methods. In the final section, we are going to illustrate the basics of sparse grids (SG) construction and SGQ formulas.

## 2.1 Pricing Theory

### 2.1.1 Financial Derivatives

Financial derivatives are securities whose value depends on the price of one or more other underlying assets, for example: stocks, stock indices, bonds, exchange rates or commodities. Financial derivatives are either traded at special derivatives exchanges in a similar way to the underlying assets or directly over-the-counter between financial institutions.

The main topic of this work is investigating the impact of the input parameters uncertainty on the fair values of such financial derivatives. This fair value does not have to be equal to the market value of the derivative which results from supply and demand and thus the subjective notions of the value of the derivative from buyers and sellers. Nevertheless, the fair value is an important notion for all market participants. Historically, mathematically well-founded fair prices which were derived by Black-Scholes and Merton [1] eventually enabled

the systematic trade of financial derivatives after the introduction of derivative exchange at the Chicago Board of Trade in 1973. Numerical methods, i.e. approximation algorithms, play a crucial rule for the valuation of financial derivatives since in almost all cases of derivatives and corresponding model assumptions no closed-form solution for their fair value can be derived.

There are many types of financial derivatives which are currently traded in the markets or which are used for the assessment and hedging of risks. The variety of these derivatives has been growing constantly in the last years. An overview is given in [4, 9] but here we just focus on European single asset and basket options.

#### 2.1.1.1 European Options

The simplest type of options are European options. Nevertheless, they are of great practical (and theoretical) importance. As a European Option is a standard option we should start with defining this family of options.

**Definition 2.1.1. ( Standard Option)** *A standard (vanilla) option bears the right, but not the obligation, to buy or sell a certain number of the underlying securities for a prescribed price within a certain time period. Options which allow the holder to buy the underlying securities are called call options, while options which include the right to sell them are called put options. The prescribed price $K$ is often called strike or exercise price and the time in which the option can be exercised is called exercise time or exercise period.*

**Definition 2.1.2. (European Option)** *A European option is a standard option where the exercise period consists of a single point in time in the future, the exercise time $T > 0$.*

**Definition 2.1.3. (Payoff of Standard Options)** *The value of a European call option at the exercise time $T$ is given by the payoff*

$$V(S,T) := (S-K)^+ := \max\{\text{S-K}, 0\} \tag{2.1}$$

*The value of a European put option at the exercise time is correspondingly*

$$V(S,T) := (K-S)^+ := \max\{\text{K-S}, 0\} \tag{2.2}$$

When computing option prices one can, at least for European options, confine oneself either to call or to put options since the so-called put-call parity holds.

$$S(t) + V_{Put}(S,t) = V_{Call}(S,t) + Ke^{-r(T-t)}. \tag{2.3}$$

Here, r is the riskless interest rate, i.e. the interest a riskless investment generates, which is assumed to be constant over time.

#### 2.1.1.2 Basket Options

A basket option is a multi-asset option which means that it is written on two or more underlyings.
We assume that there are n assets involved in total. The price of the $i-th$ asset varying with time $t$ is denoted by $S_i(t)$ , $1 \leq i \leq n$. All asset prices at the end of the exercise time $t = T$ are collected in the vector $S_T = (S_1(T), ..., S_n(T))$.

**Definition 2.1.4. (Basket Option payoff)** *For an arithmetic average basket call option, the payoff reads*

$$V(S,T) = \left( \frac{1}{n} \sum_{i=1}^{n} S_i - K \right)^+, \tag{2.4}$$

*while for a geometric average, the payoff is given by*

$$V(S,T) = \left( \left( \prod_{i=1}^{n} S_i \right)^{1/n} - K) \right)^+. \tag{2.5}$$

*For basket put options, the roles of the average and the strike are reversed.*

Weighted averages are also often used, especially in the arithmetic average. Thereby, in the summation, each asset price is multiplied with a weight $\omega_i$, $1 \leq i \leq n$ with $\sum_{i=1}^{n} \omega_i = 1$ indicating the importance of the asset in the basket.

### 2.1.2 Stochastic Market Models

#### 2.1.2.1 Introduction

In the following, we will take a look at often used models for the future development of single as well as multiple interacting asset prices, in particular so-called Black-Scholes models. In the univariate case, we will consider two methods for the determination of the most important parameter in this model, the volatility. First, we have to secure a few important market assumptions.

#### 2.1.2.2 Market Assumptions

The following assumptions on the market are usually made:
- There are no transaction costs or taxes.
- The interest rates for loaning and lending are equal and constant for all parties.

- All parties have access to all information.
- Securities and credits are available at any time and in any quantity.
- Short sales are permitted.
- The individual trade does not influence the price.
- There are no arbitrage opportunities.

The first few assumptions are made only for simplification purposes and can later be suspended or suitably modeled. Especially the last assumption of absence of arbitrage is of central importance for the fair valuation of financial derivatives, though.

#### 2.1.2.3 Black-Scholes Model

One of the most basic stochastic models for stocks was developed by Bachelier about 1900 [10]. This model is still used today also for other types of securities. It is the foundation of the pioneering works of Black-Scholes and Merton [1] on option pricing. In the Black-Scholes model, the future development of the underlying is modeled by means of a geometric Brownian motion and follows a linear stochastic differential equation (SDE).

**Definition 2.1.5. (Black-Scholes Model)** *The Black-Scholes model for a single underlying asset is given by the SDE*

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \tag{2.6}$$

*where $\mu$ represents the constant drift, $\sigma$ the constant volatility and $W(t)$ a one-dimensional Wiener process (standard Brownian motion).*

A Wiener process is a Markov process with properties $W(0) = 0$ and $W(t) \sim \mathcal{N}(0, t)$ for $t > 0$. Thereby, $\mathcal{N}(0, t)$ is the Gaussian normal distribution with mean 0 and variance $t$. Above notation is just an abbreviated form for the Itô integral equation

$$S(t) = S(0) + \int_0^t \mu S(u)du + \int_0^t \sigma S(u)dW(u) \tag{2.7}$$

For this integral equation there exists a closed-form solution as

$$S(t) = S(0)e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}, \tag{2.8}$$

which can be shown via Itô's lemma.

For option pricing, the stochastic process has to be transformed into its risk-neutral form. In the Black-Scholes model, only the drift $\mu$ has to be replaced by the riskless interest rate

$r$. This way, the explicit solution becomes

$$S(t) = S(0)e^{(r-\frac{\sigma^2}{2})t+\sigma W(t)}. \tag{2.9}$$

Dividing by $S(0)$ and taking the logarithm of both sides results in

$$\ln(S(t)/S(0) = (r - \frac{\sigma^2}{2})t + \sigma W(t). \tag{2.10}$$

This way, one can see that the value increment $S(t)/S(0)$ is normally distributed with mean 0 and variance $t$ and thus $S(t)$ is log-normally distributed.

#### 2.1.2.4 Further Single-Asset Models

The Black-Scholes model is by no means the only stochastic model which is used to describe the future development of assets (see [11]). One point of criticism is that the Black-Scholes model does not properly reflect the dependence of the option price on the volatility. This lead to the idea that the volatility should follow its own stochastic process.

**Definition 2.1.6. (Stochastic Volatility Model)** *In the stochastic volatility model the asset price dynamics are given by the system of SDEs*

$$dS(t) = \mu S(t)dt + \sigma(t)S(t)dW(t). \tag{2.11}$$

$$d\sigma(t) = a\sigma(t)dt + b\sigma(t)d\widetilde{W}(t). \tag{2.12}$$

*with some constants $a$, $b$ and with $W(t)$ and $\widetilde{W}(t)$ being two Wiener process with correlation $\rho$, that is* $\mathrm{E}[W(t)\tilde{W}(t)] = \rho t$.

Stochastic volatility models have the disadvantage that three more parameters ($a$, $b$ and $\rho$) have to be estimated from market data. Also, they are more difficult to simulate than the Black-Scholes model since no closed-form solution of the system is known.

Another critical point of the Black-Scholes model is that it underestimates extreme up and downward movements of many assets, such as stocks. This problem can be removed by using more heavy-tailed distributions for the random increments. Popular examples are jump-diffusion models where extreme events are modeled by jumps of the underlying. To this end, additional jump term is added to the Black-Scholes model.

**Definition 2.1.7. (Jump-Diffusion Model)** *In a jump-diffusion model, the asset price follows the SDE*

$$dS = \mu S dt + \sigma S dW + \eta S dN, \tag{2.13}$$

*where $N$ is a Poisson process with intensity $\lambda$, i.e.*

$$dN = \begin{cases} 0 & \text{with probability } 1 - \lambda dt \\ 1 & \text{with probability } \lambda dt \end{cases}$$

*and $\eta$ is an impulse function which generates a jump from $S$ to $S(1+\eta)$.*

Many forms of $\eta$ such as normal, singular or hypersingular distributions have been proposed in the literature. Jump-diffusion models, however, have as disadvantage that they result in an incomplete which makes option pricing by martingale methods much more difficult.

#### 2.1.2.5 Parameter Estimation

In the Black-Scholes model, two parameters occur, the drift $\mu$ and the volatility $\sigma$ which have to be determined from market data. As we have seen, the drift does not occur in the risk-neutral form of the stochastic differential equation, the volatility plays a very important role, however. We will now consider two methods for volatility estimation.

**Historical volatility** One possibility for the determination of the volatility consists in the observation of past prices of the underlying. This historical volatility corresponds to the variance of the logarithmic prices over past times. Let $t_k$, $0 \leq k \leq n$, be $n+1$ points in time and $S(t_k)$ the prices of the underlying at these times. Since the prices are log-normally distributed in the Black-Scholes model, the historical volatility can be computed by

$$\sigma^2 = \frac{1}{n-1} \sum_{j=1}^{n} (\ln(S(t_j)/S(t_{j-1})) - \bar{S})^2, \tag{2.14}$$

where

$$\bar{S} = \frac{1}{n-1} \sum_{j=1}^{n} \ln(S(t_j)/S(t_{j-1})) \tag{2.15}$$

**Implied volatility** Alternatively, the volatility can be computed from the market price of other options on the same underlying. This method is often used since in the Black-Scholes model actually the future and not the past volatility has to be used. The volatility implied by the marked is for trading purposes even more important than the option price itself. If an algorithm for approximation of option prices with varying volatility and its Vega ($\Lambda = \frac{\partial V}{\partial \sigma}$) is known, the implied volatility can be computed by iterative zero finding, e.g. using the Newton-Raphson method.

$$\sigma_{j+1} = \sigma_j - \frac{V(\sigma_j) - V}{\Lambda(\sigma_j)}, \tag{2.16}$$

starting with an estimated volatility of $\sigma_0$. Here, $V(\sigma_j)$ is the option price for the iterate the corresponding Vega and $V$ the market price of the option.

#### 2.1.2.6 Multi-Asset Models

Now, we consider some generalizations of the Black-Scholes model for several interacting assets, see [12–14]. To this end, again systems of stochastic differential equations are used. Here, we discern between two cases. In the so-called full model, the number of stochastic processes equals the number of assets while in the so-called reduced model, the number of stochastic processes is smaller. In both cases, the resulting markets are complete, only if the number of stochastic processes is larger than the number of assets, the market would become incomplete [11].

**Black-Scholes Model**

We start with the full Black-Scholes model where the number of stochastic processes equals the number of assets $n$.

**Definition 2.1.8. (Full Black-Scholes Model)** *In the full multivariate Black-Scholes model, the asset price dynamics of n assets is given by the system of SDEs*

$$dS_i(t) = S_i(t)\left(\mu_i dt + \sum_{j=1}^{n} \rho_{ij} dW_j(t)\right) \tag{2.17}$$

*for $i = 1, ..., n$, where $\mu_i$ denotes the drift of the $i-th$ stock, $(\rho_{ij})_{1\leq i\leq n, 1\leq j\leq n}$ is the $n \times n$ correlation matrix of the stocks' prices movements and $W_j(t), 1 \leq j \leq n$ , Brownian motions.*

The matrix $\rho\rho^T$ is assumed to be strictly positive definite. The explicit solution of the system of SDEs (2.17) is given by

$$S_i(T) = S_i(X) = S_i(0)exp\left(\mu_i T - \bar{\rho}_i + \sqrt{T}\sum_{j=1}^{n} \rho_{ij} X_j\right) \tag{2.18}$$

for $i = 1, .....n$ with

$$\bar{\rho}_i := \frac{1}{2}\sum_{j=1}^{n} \rho_{ij}^2 T \tag{2.19}$$

and $X = (X_1, ...., X_n)$ being a $\mathcal{N}(0, I)$-normally distributed random vector.
The full Black-Scholes model is typically used if the number of assets is small. The entries of the correlation matrix can be estimated efficiently based on historical data. To this end, the covariance of the logarithmic prices is estimated.

**Reduced Black-Scholes Model** For a larger number of assets, however, the parameter estimation problem can become more and more ill-conditioned resulting in eigenvalues of $\rho\rho^T$ which are close to zero. In this case, so-called reduced Black-Scholes models are typically used. There, it is assumed that the asset price movements are driven by $d < n$ stochastic processes.

### 2.1.3 Pricing Approaches

#### 2.1.3.1 Introduction

The prices of financial derivatives depend on the expected future development of the underlying assets. This development is presumed to be given by a stochastic differential equation or a system of equations some of which were illustrated in the previous chapter. Under these models and market assumptions, formulas for the fair prices of financial derivatives can be mathematically derived which is the subject of this chapter.

This fair price is usually given as an expectation or the solution of a partial differential equation. The connection between these representations shows the Feynman-Kac theorem (see [15]). In both cases, the price of the derivative can be computed after suitable discretization (in space and time) and solution of the resulting discrete problem (see Figure2.1). In the first case, an integration problem has to be computed, in the second case a large linear system has to be solved. For a fast and accurate computation of derivative prices, special numerical methods have to be used in these discretization and solution steps.

In the following, we do not follow the PDE approach but consider only the martingale approach that we will use in the course of this work and which corresponds to the left branch in the tree of Figure 2.1.

#### 2.1.3.2 Pricing Principles

The following three main principles from the mathematical theory of derivatives pricing are important here, see [16]

1. If a derivative security can be perfectly replicated( hedged) through trading in other assets, then the price of the derivative security is the cost of the replicating trading strategy.
2. Discounted asset prices are martingales under a probability measure associated with the choice of numeraire. Prices are expectations of discounted payoffs under such a martingale measure.

3. In a complete market, any payoff(satisfying modest regularity conditions) can be synthesized through a trading strategy, and the martingale measure associated with a numeraire is unique. In an incomplete market there are derivative securities that cannot be perfectly hedged; the price of such a derivative is not completely determined by the prices of other assets.

The first principle tells us what the price of a derivative security ought to be but does not show us how this price can be evaluated. The second principle tells us how to represent prices as expectations. The third principle states under what conditions the price of a derivative security is determined by the prices of other assets so that the first and second principles are applied.

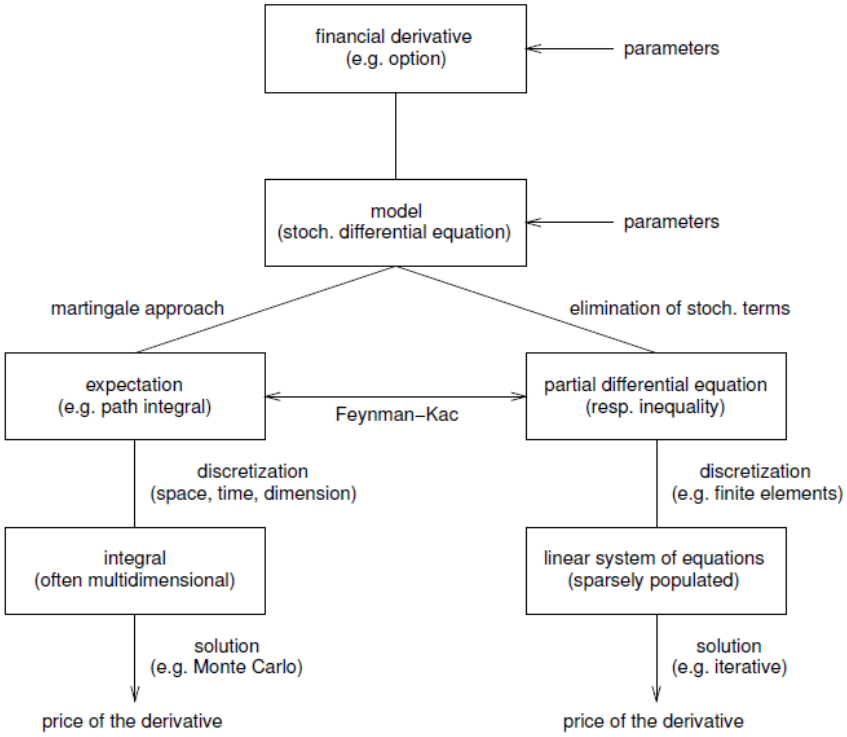



Figure 2.1: Pricing approach

#### 2.1.3.3 Martingale Approach

The martingale approach is one of the main principles for option pricing. It says that the fair price of an option is the discounted expectation of the payoff under the risk neutral probability distribution of the underlying economic factors.

### Standard Options

The martingale representation of fair financial derivative prices has been found much later than the pioneering works of Black-Scholes and Merton which are based on the PDE representation.

**Theorem 2.1.1.** ***(Fair Value of European Options)*** *The fair value of a European call option under the Black-Scholes model is given by*

$$V(S,0) = e^{-rT} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \left( S(0) e^{(r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}x} - K \right)^+ dx. \tag{2.20}$$

*For a European put option, the difference is simply reversed.*

**Proof 2.1.2.** *see [12].*

### Multi-Asset Options

The multi-asset options we have considered are of European type in the sense that they can be exercised only at the exercise time $T$. The full and reduced multivariate Black-Scholes models induce a complete market which gives the existence of a unique equivalent martingale measure, see [13]. Under this measure, all drifts $\mu_i$ in (2.18) are replaced by the riskless interest rate $r$ for each asset. This way, we have the following representation for the fair value of multi-asset options.

**Theorem 2.1.3.** ***(Fair Value of Multi-Asset Options)*** *The fair value of a (European style) multi-asset option is under the full or the reduced multivariate Black-Scholes model given by*

$$V(S,0) = e^{-rT} \int_{\mathbb{R}^d} \phi(X) V(S(X),T) dX \tag{2.21}$$

*where $\phi(X) := \phi_{0,I}(X)$ is the multivariate normal distribution with mean 0 and covariance matrix $I$.*

**Proof 2.1.4.** *We start with the martingale representation*

$$V(S,0) = e^{-rT} E^*[V(S(X),T)] \tag{2.22}$$

*where $E^*$ is the expectation under the equivalent martingale measure. Plugging in the density function $\phi$ of the underlying random vector $X$, we get the assertion.*
*Note that here, $d \leq n$, which incorporates both the full and the reduced Black-Scholes models.*

### 2.1.4 Valuation Formula for European Single Asset Options

The valuation formula of European options under the Black-Scholes model assumptions is known as Black-Scholes formula. Due to its simplicity, the Black-Scholes formula is of great practical importance for the trading of options.

**Theorem 2.1.5.** ***(Black-Scholes Formula)*** *In the Black-Scholes model, the fair price of a European call option is given by*

$$V(S,0) = S\Phi(d_1) - Ke^{-rT}\Phi(d_2), \tag{2.23}$$

*and of a European put option by*

$$V(S,0) = Ke^{-rT}\Phi(-d_2) - S\Phi(-d_1), \tag{2.24}$$

*where*

$$d_1 = \frac{ln(S/K) + (r + \frac{\sigma^2}{2})(T)}{\sigma\sqrt{T}},$$

$$d_2 = \frac{ln(S/K) + (r - \frac{\sigma^2}{2})(T)}{\sigma\sqrt{T}}$$

*and $\Phi$ denotes the cumulative distribution function of the standard normal distribution.*

**Proof 2.1.6.** *See [12]*

### 2.1.5 Valuation Formulas for Basket Options

#### 2.1.5.1 Introduction

As a basket option is an option whose payoff depends on the average of some particular assets, we define a basket of stocks by

$$A(T) = \sum_{i=1}^{n} \omega_i S_i(T), \tag{2.25}$$

where $A(T)$ is the weighted arithmetic average of $n$ underlying stocks. Then, the payoff of a Call($\theta = 1$) resp. Put($\theta = -1$) reads as

$$P_{Basket}(A(T), K, \theta) = [\theta(A(T) - K)]^+ \tag{2.26}$$

The problem of pricing basket options in the Black-Scholes model is the following: The stock prices are modeled by geometric Brownian motions and are therefore log-normally distributed. As the sum of log-normally distributed random variables is not log normal, it is not possible to derive an(exact) closed-form representation of the basket call and put prices.

Due to the fact that we are dealing with a multidimensional process, only Monte Carlo and over multidimensional integration methods are suitable numerical methods to determine the value of these options. As these methods can be very time consuming many alternative valuation methods were developed and which are based on analytical approximations in different senses. Among these approximations we mention:

- **Levy** (1992) uses a log-normal distribution with matching moments,
- **Gentle**(1993) approximates the arithmethic average by a geometric one,
- **Milevsky & Posner** (1998) applies the reciprocal gamma distribution,
- **Beisser** (1999) performs some conditional expectation techniques,and
- **Ju** (2002) uses Taylor expansion.

In [17] we find an analysis of the performance of the first four methods which concludes after numerical tests that the approximations of Levy and Beisser are overall the best performing methods. In our work, we will check the performance of Ju's approximation to those of Levy and Beisser. The best method will be then used to study the effect of uncertainty in high dimensional case. Therefore, we take a look here at these approximations.

#### 2.1.5.2 Ju's Taylor Expansion

Ju's approximation [18] is based on Taylor expansion of the ratio of the characteristic function of the arithmetic average to that of the approximating log-normal random variable around zero volatility.

Let consider the following standard n-asset under the risk-neutral measure

$$S_i(t) = S_i(0)e^{(g_i-\sigma_i^2/2)t+\sigma_i W_i(t)}, \quad i = 1, 2, ....., n \tag{2.27}$$

where $g_i = r - \delta_i$, $r$ is the riskless interest rate, $\delta_i$ the dividend yield, $\sigma_i$ the volatility, $W_i(t)$ a standard Wiener process. Let $\rho_{ij}$ denote the correlation coefficients between $W_i(t)$ and $W_j(t)$.
To apply the Taylor expansion all the individual volatilities are scaled by the same parameter $z$, so we have

$$S_i(z,t) = S_i(0)e^{(g_i-z^2\sigma_i^2/2)t+z\sigma_i W_i(t)}, \quad i = 1, 2, ....., n \tag{2.28}$$

and

$$A(z) = \sum_{i=1}^{n} \omega_i S_i(z, T) \tag{2.29}$$

Let $Y(z)$ be a normally distributed random variable with mean $m(z)$ and variance $v(z)$ such that the first two moments of $\exp(Y(z))$ match those of $A(z)$ . So we have

$$m(z) = 2\log U_1 - 0.5\log U_2(z^2) \tag{2.30}$$

$$v(z) = \log U_2(z^2) - 2\log U_1, \tag{2.31}$$

where $U_1$ and $U_2$ are respectively the first and second moment of $A(z)$ given by(2.32) and(2.33)

$$U_1 = \sum_{i=1}^{n} \bar{S}_i = A(0) \tag{2.32}$$

$$U_2(z^2) = \sum_{ij=1}^{n} \bar{S}_i \bar{S}_j e^{z^2 \bar{\rho}_{ij}} \tag{2.33}$$

with

$$\bar{S}_i = \omega_i S_i(0) e^{g_i T} \; and \; \bar{\rho}_{ij} = \rho_{ij}\sigma_i\sigma_j T$$

If we define $X(z) = \ln(A(z))$, then

$$E[e^{itX(z)}] = E[e^{itY(z)}]\frac{E[e^{itX(z)}]}{E[e^{itY(z)}]} = E[e^{itY(z)}]f(z), \tag{2.34}$$

where

$$E[e^{itY(z)}] = e^{itm(z) - t^2 v(z)/2} \tag{2.35}$$

is the characteristic function of the normal random variable and

$$f(z) = \frac{E[e^{itX(z)}]}{E[e^{itY(z)}]} = E[e^{itX(z)}]e^{-itm(z) + t^2 v(z)/2} \tag{2.36}$$

is the ratio of the characteristic function of $X(z)$ to that of $Y(z)$.
Ju performs a Taylor expansion of the two factors of $f(z)$ up to $z^6$, leading to

$$f(z) \approx 1 - itd_1(z) - t^2 d_2(z) + it^3 d_3(z) + t^4 d_4(z) \tag{2.37}$$

where $d_i(z)$ are polynomials of $z$ and terms of higher order then $z^6$ are ignored. Finally $E[e^{itX(1)}]$ is approximated by

$$E[e^{itX(1)}] \approx e^{itm(1) - t^2 v(1)/2}(1 - itd_1(1) - t^2 d_2(1) + it^3 d_3(1) + t^4 d_4(1)) \tag{2.38}$$

For this approximation, an approximation of the density $h(x)$ of $X(1)$ is derived as

$$h(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}e^{itm(1)-t^2v(1)/2}(1 - itd_1(1) - t^2d_2(1) + it^3d_3(1) + t^4d_4(1))dt \quad (2.39)$$

$$= p(x) + (d_1(1)\frac{d}{dx} + d_2(1)\frac{d^2}{dx^2} + d_3(1)\frac{d^3}{dx^3} + d_4(1)\frac{d^4}{dx^4})p(x) \quad (2.40)$$

where $p(x)$ is the normal density with mean $m(1)$ and variance $v(1)$. The approximate price of a basket call is then given by,

$$V_{Basket\,Call} = e^{-rT}E[e^{X(1)} - K]^+ \quad (2.41)$$

$$= [U_1e^{-rT}\Phi(y_1) - Ke^{-rT}\Phi(y_2)] + [e^{-rT}K(z_1p(y) + z_2\frac{dp(y)}{dy} + z_3\frac{d^2p(y)}{dy^2}], \quad (2.42)$$

where

$$y = \ln(K)\ ,\ y_1 = \frac{m(1) - y}{\sqrt{v(1)}} + \sqrt{v(1)}\ ,\ y_2 = y_1 - \sqrt{v(1)},$$

and

$$z_1 = d_2(1) - d_3(1) + d_4(1)\ ,\ z_2 = d_3(1) - d_4(1)\ ,\ z_3 = d_4(1).$$

#### 2.1.5.3 Levy's Approximation

The basic idea of Levy's approximatio [19] is to approximate the distribution of the basket by a log-normal distribution with mean $\bar{M}$ and variance $\bar{V}^2$. These two parameters are determined in such a way that they match the true moments of the arithmetic average $A(T) = \sum_{i=1}^{n}\omega_iS_i(T)$, i.e.

$$E(A(T)) = E(e^X) \quad and \quad Var(A(T)) = Var(e^X)$$

where $X$ is a normally distributed random variable with mean $\bar{M}$ and variance $\bar{V}^2$ . The basket option price can now be evaluated as

$$V_{Basket\,Call}(T) = e^{-rT}\left(e^{\bar{M}+\frac{1}{2}\bar{V}^2}\Phi(d_1) - K\Phi(d_2)\right) \quad (2.43)$$

where $\Phi$ is the distribution function of a standard normal random variable and

$$d_1 = \frac{\bar{M} - \ln K + \bar{V}^2}{\bar{V}^2},$$

$$d_2 = d_1 - \bar{V}$$

.

#### 2.1.5.4 Beisser's Approximation

The main idea behind the method of Beisser is the conditional expectation technique introduced by Rogers & Shi (1995) [20] for the pricing of Asian Options. More precisely, to condition the payoff 's expectation with a normal distributed random variable $Z$ using the tower law, then estimate the result by applying Jensen's inequality and derive a closed-form solution for this estimate. In a nutshell this reads as

$$\begin{aligned} \mathrm{E}\left(\mathrm{E}\left[A(T)-K\right]^{+}\right) &= \mathrm{E}\left(\mathrm{E}\left(\left[A(T)-K\right]^{+} \mid Z\right)\right) && (2.44)\\ &\geq \mathrm{E}\left(\mathrm{E}\left(\left[A(T)-K \mid Z\right]^{+}\right)\right) && (2.45)\\ &= \mathrm{E}\left(\left[\sum_{i=1}^{n}\omega_i\,\mathrm{E}[S_i(T) \mid Z]-K\right]^{+}\right) && (2.46) \end{aligned}$$

where

$$Z := \frac{\sigma_z}{\sqrt{T}}W(T) = \sum_{i=1}^{n}\omega_i S_i(0)\sigma_i W_i(T) \tag{2.47}$$

with $\sigma_Z$ appropriately chosen. Since all conditional expectations $E[S_i(T)|Z]$ are log-normally distributed with respect to one Brownian motion W(T). There exists an $x^*$, such that

$$\sum_{i=1}^{n}\omega_i\,\mathrm{E}\left[S_i(T) \mid W(T)=x^*\right] = K. \tag{2.48}$$

By defining

$$\tilde{K}_i := \mathrm{E}\left[S_i(T) \mid W(T)=x^*\right]$$

the event $\sum_{i=1}^{n}\omega_i\,\mathrm{E}\left[S_i(T) \mid Z\right] \geq K$ is equivalent to $\mathrm{E}\left[S_i(T) \mid Z\right] \geq \tilde{K}_i$ for all $i = 1.....n$. Using this equivalence we conclude that

$$\begin{aligned} \mathrm{E}\left(\left[\sum_{i=1}^{n}\omega_i\,\mathrm{E}[S_i(T) \mid Z]-K\right]^{+}\right) &= \sum_{i=1}^{n}\omega_i\,\mathrm{E}\left(\left[\mathrm{E}\left[S_i(T) \mid Z\right]-\tilde{K}_i\right]^{+}\right) && (2.49)\\ &= \sum_{i=1}^{n}\omega_i\left[\tilde{F}_i^{T}\Phi(d_{1i})-\tilde{K}_i\Phi(d_{2i})\right] && (2.50) \end{aligned}$$

where $\tilde{F}_i, \tilde{K}_i$ adjusted forwards and strikes and $d_{1i}$, $d_{2i}$ are the usual terms with modified parameters( see [20]).

## 2.2 Uncertainty Quantification (UQ)

### 2.2.1 Introduction

The importance of understanding uncertainty has been realized by many for a long time in disciplines such as civil engineering, hydrology, control,mechanics etc. Consequently many methods have been devised to tackle this issue. Because of the "uncertain" nature of the uncertainty, the most dominant approach is to treat data uncertainty as random variables or random processes and recast the original deterministic systems as stochastic systems.

As we mentioned in 1.2.2, UQ in finance began with the use of deterministic methods which are mainly sensitivity analysis and worst case methods but recently probabilistic methods have gained popularity over deterministic methods in many fields. In fact, probabilistic methods provide additional information which is the distribution of the model 's output. The cost is that one must specify a distribution for the input parameter. Given the distribution, one may then compute statistics, e.g. mean and standard deviation, or $\alpha$ bounds.

Probabilistic uncertainty analysis allows one to model the uncertainty from the information available. Once one specifies a distribution for the input, one tacitly indicates how much information is at hand. If very little information is available, one might model the input by a uniform distribution. On the other extreme, if the input is known exactly, the situation corresponds to a point mass, or the Dirac-Delta distribution. More likely though, the information available is somewhere in between, in which case a normal density might be used.

In this section, we review the probabilistic methods for uncertainty analysis that we are going to use in the course of our project which are MC and PC methods.

### 2.2.2 Monte Carlo Technique

The method of Monte-Carlo simulation is a numerical technique used to solve mathematical problems by simulating random variables. There is no absolute consensus on a precise definition of a MC technique, but the most common description is that the methods of this type are characterized by the generation of random samples to solve problems centered on probabilistic calculations. In finance, MC simulation [16] is often used to approximate a complex expectation. For example, to approximate $\mathrm{E}[f(Y)]$ where $Y$ is a random variable we follow algorithm1.

**Algorithm 1** Monte Carlo Simulation

**Step** 1 Simulate a great number $N$ of independent random variables $\{Y_n\}_{1\leq n\leq N}$ having the same distribution as $Y$

**Step** 2 For each $n = 1, ....., N$ calculate $f(Y_n)$

**Step** 3 Calculate $\frac{1}{N}\sum_{n=1}^{N} f(Y_n)$ as an approximation to $\mathrm{E}[f(Y)]$

The convergence is ensured by the following classic theorems:

**Theorem 2.2.1.** ***(Law of Large Numbers)*** *Let $(Y_n)_{n\geq 1}$ be a sequence of integrable independent random variables, all having the same distribution as a variable $Y$. We have then:*

$$\frac{1}{N}\sum_{n=1}^{N} Y_n \underset{N\to+\infty}{\longrightarrow} \mathrm{E}[Y] \quad a.s. \tag{2.51}$$

**Theorem 2.2.2.** ***(Central Limit Theorem)*** *Let $(Y_n)_{n\geq 1}$ be a sequence of integrable independent random variables, all having the same distribution as a variable $Y$. We define the empirical mean and variance by*

$$\hat{m}_N := \frac{1}{N}\sum_{n=1}^{N} Y_n,$$

$$\hat{\sigma}_N = \sqrt{\frac{1}{N-1}\sum_{n=1}^{N}(Y_n - \hat{m}_N)^2}.$$

*We have then :*

$$\sqrt{N}\left(\frac{\hat{m}_N - \mathrm{E}[Y]}{\hat{\sigma}_N}\right) \underset{N\to+\infty}{\overset{in\ law}{\longrightarrow}} \mathcal{N}(0,1) \tag{2.52}$$

The second theorem allows us to build a confidence interval for $\mathrm{E}[Y]$. In fact, when $N$ is large enough, we can write for a positive real $c$:

$$P\left(\mid \frac{\hat{m}_N - \mathrm{E}[Y]}{\hat{\sigma}_N} \mid < c\right) = 1 - \alpha_c,$$

where $\alpha_c = P(\mid X \mid > c)$ with X having a normal distribution is the significance level (we

put in general $\alpha_c = 5\%$). We have then:

$$P\left(\mathrm{E}[Y] \in \left[\hat{m}_N - c\frac{\hat{\sigma}_N}{\sqrt{N}}, \hat{m}_N + c\frac{\hat{\sigma}_N}{\sqrt{N}}\right]\right) = 1 - \alpha_c. \tag{2.53}$$

$\left[\hat{m}_N - c\frac{\hat{\sigma}_N}{\sqrt{N}}, \hat{m}_N + c\frac{\hat{\sigma}_N}{\sqrt{N}}\right]$ is the confidence interval associated with the significance level $\alpha_c$.

MC method is easy to implement, but needs a huge number of scenarios to give a good accuracy. It is a method known to be very heavy in calculations, taking in general much CPU-time to converge. To improve the convergence, we can increase the number of samples, which is not a clever solution because the method will take then much more time to give a result, or try to decrease the variance $\hat{\sigma}^2_N$. The second solution is known as Variance Reduction Techniques. Although variance reduction methods are not so complicated, their application to financial simulation problems is not trivial. In fact to design a successful variance reduction method, one has to understand the characteristics of the problem of interest (see [16]). In our work, we are going to use The Control Variate(CV) technique as a variance reduction method. Therefore, we are going to give in the next section basic information about this technique.

### 2.2.3 Control Variate (CV) Technique

CV method is the most effective and broadly applicable technique for variance reduction of simulation estimates. [16] defines the method as a way of exploiting the information about the errors in estimates of known quantities to reduce the error in an estimate of an unknown quantity. To describe the method, let's suppose that we wish to estimate $\mu = \mathrm{E}[X]$, where X is the output of a simulation experiment. Suppose that $Y$ is also an output of the simulation or that we can easily output it if we wish. Finally, we assume that we know $\mathrm{E}[Y]$. Then we can construct many unbiased estimators of $\mu$.

- $\hat{\mu} = X$, our usual estimator.
- $\hat{\mu}_c = X - c(Y - \mathrm{E}[Y])$ , where c is some real number.

Here $Y$ is called a control variate for $X$ . It is clear that $\mathrm{E}[\hat{\mu}_c] = \mu$. The question is whether or not $\hat{\mu}_c$ has a lower variance than $\mu$ . To answer this question, we compute $\mathrm{Var}(\hat{\mu}_c)$ and get:

$$\mathrm{Var}(\hat{\mu}_c) = \mathrm{Var}(X) + c^2\,\mathrm{Var}(Y) - 2c\,\mathrm{Cov}(X, Y). \tag{2.54}$$

Since we are free to choose $c$, we should choose it to minimize $\mathrm{Var}(\hat{\mu}_c)$. Simple calculus then implies that the optimal value of $c$ is given by

$$c^* = \frac{\mathrm{Cov}(X, Y)}{\mathrm{Var}(Y)}. \tag{2.55}$$

Substituting for $c^*$ into the variance formula above we see that

$$\mathrm{Var}(\hat{\mu}_{c^*}) = \mathrm{Var}(X) - \frac{\mathrm{Cov}(X,Y)^2}{\mathrm{Var}(Y)} \tag{2.56}$$

$$= \mathrm{Var}(X) - \frac{\mathrm{Var}(X)\,\mathrm{Var}(Y)\rho_{XY}^2}{\mathrm{Var}(Y)} \tag{2.57}$$

$$= \mathrm{Var}(X)(1-\rho_{XY}^2), \tag{2.58}$$

where $\rho_{XY}$ denotes the correlation between X and Y . The above formula shows that the CV method is successful only if the selected CV is highly correlated with the original simulation output.

### 2.2.4 Polynomial Chaos (PC) Method

#### 2.2.4.1 Introduction

The basic idea of Polynomial Chaos is to compose orthogonal polynomials with independent random variables. The benefit is that it reduces the problem of constructing a basis on a probability space to the problem of constructing orthogonal polynomials with respect to a given weight function. The original polynomial chaos [21, 22] employs the Hermite polynomials in the random space as the trial basis to expand the stochastic processes. Cameron and Martin proved that such expansion converges to any second-order processes in the $L_2$ sense [23].

Ghanem provided a survey of the subject in 1991 [24] and restricted his attention to Hermite polynomials composed with Gaussian random variables as his basis. Xiu and Karniadakis [25] introduced generalized PC, which generalized the method to include other polynomials and random variable families. In 2010, Xiu [26] published a text detailing the use of PC in stochastic spectral methods.

The chaos expansion is essentially a representation of a function $f \in L_2\,(\Omega)$ where $\Omega$ is the properly defined probability space. We denote by $\{\Phi_k\}_0^\infty$ a family of polynomials that are orthogonal with respect to the density of the random variable and which form an orthogonal basis in $L_2(\mathbb{R}^n)$.

In this section, we develop the basics of PC. We recall the main mathematical notions needed to understand the present report. For more details see [27].

#### 2.2.4.2 Basic Notions and Notations

A general polynomial of degree n takes the form

$$Q_n(x) = a_n x^n + a_{n-1}x^{n-1} + ..... + a_1 x + a_0,\ a_n \neq 0 \tag{2.59}$$

where $a_n$ is the leading coefficient of the polynomial.
A system of polynomials $\{Q_n(x), n \in \mathbb{N}\}$ is an orthogonal system of polynomials with respect to some real positive measure $\alpha$ if the following orthogonality relations hold:

$$\int_S Q_n(x)Q_m(x)d\alpha(x) = \gamma_n \delta_{mn}, m, n \in \mathbb{N}, \tag{2.60}$$

where S is the support of the measure $\alpha$, and $\gamma_n$ are positive constants.
The measure $\alpha$ usually has a density $\omega(x)$ or is a discrete measure with weight $\omega_i$ at the points $x_i$. The relation (2.60) then becomes

$$\int_S Q_n(x)Q_m(x)\omega(x)dx = \gamma_n \delta_{mn},\ m, n \in \mathbb{N} \tag{2.61}$$

in the former case and

$$\sum_i Q_n(x_i)Q_m(x_i)\omega_i = \gamma_n \delta_{mn},\ m, n \in \mathbb{N} \tag{2.62}$$

in the latter case.

If we define a weighted inner product then in the continuous cases it takes the form

$$< u, v >_{d\alpha} = \int_S u(x)v(x)d\alpha(x) \tag{2.63}$$

then the orthogonality relations can be written as

$$< Q_n, Q_m >_\omega = \gamma_n \delta_{mn},\ m, n \in \mathbb{N} \tag{2.64}$$

where

$$\gamma_n =< Q_n, Q_n >_\omega = \| Q_n \|^2_\omega,\ n \in \mathbb{N} \tag{2.65}$$

#### 2.2.4.3 Polynomial Basis

PC expansion uses the Wiener-Askey scheme [25] in which Hermite, Legendre, Laguerre, Jacobi, and generalized Laguerre orthogonal polynomials are used for modeling the effect of uncertain variables described by normal, uniform, exponential, beta, and gamma probability

distributions, respectively (orthogonal polynomial selections also exist for discrete distributions, but are not explored here). These orthogonal polynomial selections are optimal for these distributions since the inner product weighting function and its corresponding support range correspond to the probability density functions for these continuous distributions. In theory, exponential convergence rates can be obtained with the optimal basis (see [25,28,29]).

Table 2.1 shows the set of polynomials which provide an optimal basis for different continuous probability distribution types. It is derived from the family of hypergeometric orthogonal polynomials known as the Askey scheme [29], for which the Hermite polynomials originally employed by Wiener [21] are a subset. The optimality of these basis selections derives from their orthogonality with respect to weighting functions that correspond to the probability density functions (PDFs) of the continuous distributions when placed in a standard form. The density and weighting functions differ by a constant factor due to the requirement that the integral of the PDF over the support range is one.

| Distribution | Density function | Polynomial | Weight function | Support range |
|---|---|---|---|---|
| Normal | $\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}$ | Hermite $e_n(x)$ | $e^{\frac{-x^2}{2}}$ | $[-\infty,+\infty]$ |
| Uniform | $\frac{1}{2}$ | Legendre $P_n$(x ) | 1 | [-1, 1] |
| Beta | $\frac{(1-x)^\alpha(1+x)^\beta}{2^{\alpha+\beta+1}B(\alpha+1,\beta+1)}$ | Jacobi $P_n(\alpha,\beta)(x)$ | $(1-x)^\alpha(1+x)^\beta$ | [-1 ,1] |
| Exponential | $e^{-x}$ | Laguerre $L_n$(x) | $e^{-x}$ | $[0,+\infty]$ |
| Gamma | $\frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)}$ | Generalized Laguerre$L_n^\alpha(x))$ | $x^\alpha e^{-x}$ | $[0,+\infty]$ |

Table 2.1: Linkage between standard forms of continuous probability distributions and Askey scheme of continuous hyper-geometric polynomials.

Where B(a,b) is the Beta function defined as $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

Since each type of polynomials form a complete basis in the Hilbert space determined by their corresponding support, we have the convergence to any $L_2$ functional in the $L_2$ sense in the corresponding Hilbert functional space as a generalized result of the Cameron-Martin theorem [28, 29]. Each type of orthogonal polynomials has weighting functions of the same form as the probability function of its associated random variables $\xi$, as shown in table 2.1.

#### 2.2.4.4 Generalized Polynomial Chaos Expansion

A general second-order random process $X(\omega)$ can be represented in the form

$$\begin{aligned} X(\omega) &= c_0\Psi_0 \\ &+ \left(\sum_{i_1=1}^{\infty} c_{i_1}\Psi_1(\xi_{i_1}(\theta))\right) \\ &+ \left(\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1} c_{i_1 i_2}\Psi_2(\xi_{i_1}(\theta),\xi_{i_2}(\theta))\right) \\ &+ \left(\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} c_{i_1 i_2 i_3}\Psi_3(\xi_{i_1}(\theta),\xi_{i_2}(\theta),\xi_{i_3}(\theta))\right) \\ &+ \cdots \end{aligned} \tag{2.66}$$

where $\Psi_n(\xi_{i1},\xi_{i2},.....,\xi_{in})$ denotes the generalized polynomial chaos of order $n$ in terms of the multidimensional random variables $(\xi_{i1},\xi_{i2},...,\xi_{in,....})$. Note that this is an infinite summation in the infinite dimensional space of $\xi$. The expansion bases$\{\Psi_n\}$ are multidimensional hypergeometric polynomials defined as tensor-products of the corresponding one dimensional polynomials bases $\{\phi_k\}_0^\infty$. Let $\chi$ be the space of index sequences $(\alpha_1,\alpha_2,.....,\alpha_n.....) \in \mathbb{N}_0^{\mathbb{N}}$ and $n := \sum_k \alpha_k$. Then,

$$\Psi_n(\xi_1,\xi_2,...,\xi_n) = \prod_{k=1}^{n}\phi_{\alpha_k}(\xi_k). \tag{2.67}$$

This representation is convergent in the mean-square sense:

$$\lim_{p\to\infty} E\left[\left(c_0\Psi_0 + ..... + \sum_{i_1=1}^{\infty}....\sum_{i_p=1}^{i_{p-1}} c_{i_1}...c_{i_p}\Psi_p(\xi_{i_1},....,\xi_{i_p}) - X\right)^2\right] = 0. \tag{2.68}$$

By construction, chaos polynomials whose orders are greater than $p = 0$ have vanishing expectation:

$$E\left[\Psi_{p>0}\right] = 0. \tag{2.69}$$

Classically, in order to facilitate the manipulation of the PC expansion, we rely on an univocal relation between $\Psi()$ and new functional $\Phi()$. It results in a more compact expression of the random variable expansion:

$$X(\omega) = \sum_{j=0}^{\infty} a_j\Phi_j(\xi(\omega)) \tag{2.70}$$

where there is a one-to-one correspondence between the coefficients and basis functions in (2.66) and (2.70).

The family $\{\Phi_n\}$ is an orthogonal basis in $L_2(\Omega)$ with orthogonality relation

$$< \Phi_i \Phi_j >=< \Phi_i^2 > \delta_{ij} \tag{2.71}$$

where $\delta_{ij}$ is the Kronecker delta and $< ., . >$ denotes the ensemble average which is the inner product in the Hilbert space of the variables $\xi$,

$$< f(\xi) g_(\xi) >= \int f(\xi) g_(\xi) W(\xi) d\xi \tag{2.72}$$

or

$$< f(\xi) g_(\xi) >= \sum_{\xi} f(\xi) g_(\xi) W(\xi) \tag{2.73}$$

in the discrete case. Here $W(\xi)$ is the weighting function corresponding to PC basis $\{\Phi_i\}$. The coefficients $a_j$ are given by

$$a_j = \frac{< X, \Phi_j(\xi) >}{< \Phi_j(\xi), \Phi_j(\xi) >} \tag{2.74}$$

$$= \frac{\int_\Omega X \Phi_j(\xi) d\mathbb{P}}{\int_\Omega \Phi_j^2(\xi) d\mathbb{P}} \tag{2.75}$$

where each inner product involves a multidimensional integral over the support range of the weighting function. In particular, $\Omega = \Omega_1 \otimes ... \otimes \Omega_n$ with possibly unbounded intervals $\Omega_j \subset \mathbb{R}$.

In practice, one truncates the infinite expansion at a finite number of random variables and a finite expansion order

$$X(\omega) \approx \sum_{j=0}^{P} a_j \Phi_j(\xi(\omega)). \tag{2.76}$$

The total number of terms $N_t$ in an expansion of total order p involving n random variables is given by

$$N_t = 1 + P = 1 + \sum_{s}^{p} \frac{1}{s!} \prod_{r=0}^{s-1} (n + r) = \frac{(n + p)!}{n! p!} \tag{2.77}$$

#### 2.2.4.5 Stochastic Sensitivity Analysis

Stochastic expansion methods have a number of convenient analytic features that make them attractive for use within higher level analyses, such as local and global sensitivity analysis and design under uncertainty algorithms. First, moments of the response expansion are available analytically. In addition, the response expansions are readily differentiated with respect to the underlying expansion variables, and response moment expressions are readily differentiated with respect to auxiliary non probabilistic variables.

If we have equation (2.76) as approximation of X then the mean and variance of the polynomial chaos expansion are available in simple closed form:

$$\mathrm{E}[X] \approx \sum_{j=0}^{P} a_j \,\mathrm{E}[\Phi_j(\xi)] = a_0, \tag{2.78}$$

and

$$\mathrm{Var}[X] \approx \mathrm{E}[(\sum_{j=1}^{P} a_j \Phi_j(\xi))^2] \tag{2.79}$$

$$= \sum_{j=1}^{P}\sum_{k=1}^{P} a_j a_k < \Phi_j(\xi), \Phi_k(\xi) > \tag{2.80}$$

$$= \sum_{k=1}^{P} (a_k)^2 \int_{\Omega} (\Phi_k(\xi))^2 d\mathbb{P} \tag{2.81}$$

These moments are exact moments of the expansion, which converge to moments of the true response function. Higher moments are also available analytically and could be employed in moment fitting approaches (i.e., Pearson and Johnson models) in order to approximate a response PDF.

## 2.3 Sparse Grid Quadrature

### 2.3.1 Introduction

When dealing with multidimensional integrals in the case of basket options, SGQ can be a convenient technique as it offers an efficient numerical treatment of multivariate problems. This method goes (at least) back to the Russian mathematician Smolyak. It has been applied to numerical integration by several authors using the rectangle rule [30], the trapezoidal rule [31], the Clenshaw-Curtis rule [32] and the Patterson rule [33] as a one dimensional basis.

In contrast to the product approach, the convergence rate of Monte Carlo and Quasi-Monte Carlo methods does not depend on the smoothness of the problem. Thus, in general, smoother integrands are not computed more efficiently than non-smooth ones. SGQ method makes use of the integrand's smoothness.

In this approach, multivariate quadrature formulas are constructed by a combination of tensor products of univariate formulas. Of all possible combinations of one-dimensional

quadrature formulas only those are taken whose corresponding indices are contained in the unit simplex. This way, the complexity becomes

$$\epsilon(N) = O\left(N^{-s}\left(\ln N\right)^{(d-1)(s+1)}\right). \tag{2.82}$$

where $N$ is the number of nodes used in SGQ method and $s$ is the smoothness' order of the integrand.
We will now briefly illustrate the basics of SG construction and SGQ formulas. More information on this subject can be found in [33].

### 2.3.2 Construction

SG can be defined for general tensor product domains $\Omega^d \subseteq \mathbb{R}^d$. We consider the case $\Omega = [0,1]$, but most results can be generalized in a straightforward way to other domains. At the end of this section we will give remarks on the case $\Omega = \mathbb{R}$.

For a univariate function $f : [0,1] \to \mathbb{R}$ and a sequence of non-decreasing integers $m_k, k \in \mathbb{N}$, let

$$U_{m_k} f := \sum_{i=1}^{m_k} w_{i,k} f(x_{i,k}) \tag{2.83}$$

denote a sequence of univariate quadrature rules with $m_k$ points $x_{i,k}$ and weights $w_{i,k}$, which converges pointwise to $If$ for $k \to \infty$. We assume $m_1 = 1$ and $U_{m_1} f = f(1/2)$ and define the difference quadrature formulae

$$\Delta_k = U_{m_k} - U_{m_{k-1}} \quad for \quad k \geq 1 \quad with \quad U_{m_0} := 0. \tag{2.84}$$

Now let $f : [0,1]^d \to \mathbb{R}$ be a multivariate function. Then, the d-dimensional integral $If$ can be represented by the infinite telescoping sum

$$If = \sum_{k \in \mathbb{N}^d} \Delta_k f \tag{2.85}$$

which collects the products of each possible combination of the univariate difference formulae. Here, $k \in \mathbb{N}^d$ denotes a multi-index with $k_j > 0$ and

$$\begin{aligned} \Delta_k f &:= (\Delta_{k_1} \otimes .... \otimes \Delta_{k_d}) f \\ &:= \sum_{i_1=1}^{n_{k_1}} ..... \sum_{i_d=1}^{n_{k_d}} \omega_{k_1 i_1} .....\omega_{k_d i_d} f(x_{l_1 i_1}, ....., x_{l_d i_d}) \end{aligned}$$

For a given level $l \in \mathbb{N}$ , the SG method, often also denoted as Smolyak method, see[1], is

defined by

$$SG_l f = \sum_{|k|_1 \leq l+d-1} \Delta_k f \tag{2.86}$$

$$= \sum_{j=d}^{l+d-1} \sum_{|k|_1=j} \Delta_k f \tag{2.87}$$

where $\mid k \mid_1 := \sum_{j=1}^{d} k_j$.
From the set of all possible indices $k \in \mathbb{N}^d$ thus only those are considered whose $\mid \ \mid_1$ -norm is smaller than a constant. That 's why it is better than the product approach where the norm $\mid . \mid_\infty = \max\{k_1, ....., k_d\}$ is used for the selection of indices.

**Comment 2.3.1.** *SGQ method can be directly applied to the numerical computation of integrals on* $\mathbb{R}^d$ *with Gaussian weight. To this end, only the sequence of univariate quadrature rules* $U_{m_k}$ *must be replaced by quadrature formulas for functions* $f : \mathbb{R} \to \mathbb{R}$ *on unbounded domains, such as Gauss-Hermite or Genz-Keister rules.*

# Chapter 3

# UQ In Option Pricing: One Dimensional Case

## 3.1 Introduction

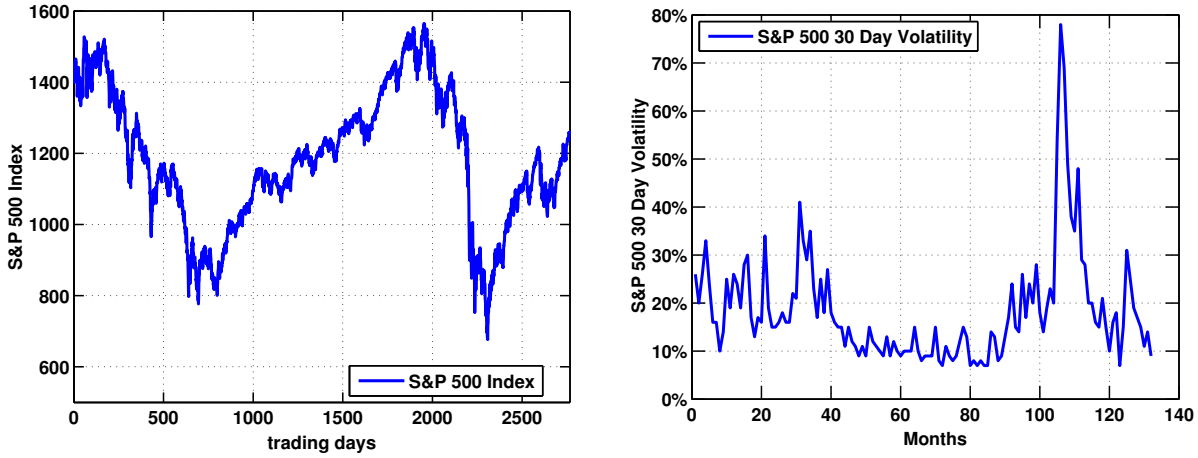
In one dimensional case, the Black-Scholes option pricing model has two parameters, the volatility $\sigma$ and risk free rate $r$, which must be specified. Uncertainty in either leads to uncertainty in the option price. In this chapter we are going to investigate the impact of volatility's uncertainty on European single asset option price under Black-Scholes framework. All numerical tests are performed with Matlab.

After parameterizing the randomness in the volatility, we present in the third section the implementation details of the probabilistic methods (MC and PC Approximation) that we use to compute the mean quantities, $\alpha$ bounds and the sample density of the put option price. Before concluding this chapter, we show the main obtained results and we compare the efficiency of crude MC versus surrogate PC/MC method.

## 3.2 Data Description and Uncertainty Parametrization

The data that we use in this chapter are S&P 500 index prices and its historical volatility. The index data source is (finance.yahoo.com) and the historical volatility was computed using a 30 day moving window (see figures 3.1a and 3.1b).

As it is shown by figure 3.1b there is uncertainty in the value of $\sigma$. The average of $\sigma$ is $\overline{\sigma} = 19\%$, its standard deviation is $stdev(\sigma) = 11\%$ and $\sigma \in [\sigma_{min} = 6\%, \sigma_{max} = 78\%]$.
Since we are using probabilistic methods for studying the relation between the uncertainty in $\sigma$ and the option price, the first step is to parametrize this uncertainty which means finding




(a) S&P 500 Index from 2000 to 2010

(b) S&P 500 historical volatility from 2000 to 2010

Figure 3.1: S&P 500 Index and historical volatility from 2000 to 2010

a distribution to represent it.

In order to find which is the best parametric probability distribution that fits to our sample of historical volatility data we tested many distributions. Our tests showed that the log-normal distribution with parameters $(-1.8, 0.5377)$ best fits the sample of data (see figure 3.2).

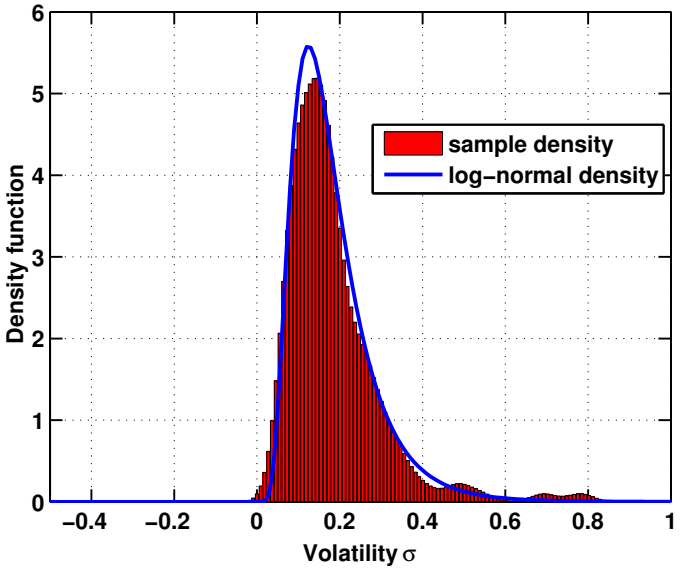



Figure 3.2: Historical volatility density

We recall that the probability density function of a log-normal distribution with parameters $\mu$ and $\sigma$ is:

$$f_X(x, \mu, \sigma) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\log x - \mu)^2}{2*\sigma^2}} \tag{3.1}$$

## 3.3 Implementation Details

### 3.3.1 MC Implementation

The input uncertainty which is in our case the volatility $\sigma$ is modeled by the distribution with density $f_\sigma$ . Then, we generate $N$ numbers $\{\sigma_i\}_{i=1}^N$ and for each $\sigma_i$, $V(S,t;\sigma_i)$ is computed to obtain $\{V_i(S,t)\}_{i=1}^N$, where $V_i(S,t) = V(S,t;\sigma_i)$. In general, one would solve Equation (3.2) for each i.

$$\frac{\partial V_i}{\partial t} = -\frac{1}{2}S^2\sigma_i^2\frac{\partial^2 V_i}{\partial S^2} - rS\frac{\partial V_i}{\partial S} + rV_i. \tag{3.2}$$

For the Vanilla put, we may use the analytical solution (3.3) to compute $\{V_i(S,t)\}_{i=1}^N$.

$$V(S,0;\sigma) = Ke^{-rT}\Phi(-d_2) - \Phi(-d_1)S, \tag{3.3}$$

where

$$d_1 = \frac{ln(S/K) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{ln(S/K) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

and $\Phi$ denotes the cumulative distribution function of the standard normal distribution.

The sample distribution of the set $\{V_i(S,t)\}_{i=1}^N$ is taken as a proxy for the true distribution of $V(S,t;\sigma)$. From the set $\{V_i(S,t)\}_{i=1}^N$, we compute the sample density of $V(S,t;\sigma)$ using the kernel density method (see [34]) , mean quantities and $\alpha$ bounds of $V(S,t;\sigma)$.

**Comment 3.3.1.** *An $\alpha$ bound for a random variable $\xi$ is the value $\xi_\alpha^+$ such that* $\text{Prob}(\xi \geq \xi_\alpha^+) = \alpha$*, i.e. an upper bound. We extend the definition to a lower bound and use the notation $-\alpha$ bound to denote the value $\xi_\alpha^-$ such that* $\text{Prob}(\xi \leq \xi_\alpha^-) = \alpha$.

### 3.3.2 PC Implementation

#### 3.3.2.1 PC Basis

As the parametric density $f_\sigma$ of the volatility is a log-normal distribution then the polynomial family that is orthogonal with respect to the volatility's density is the Hermite Polynomial (see [26]). In fact, let $Y = e^X$, where $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Z \sim \mathcal{N}(0,1)$ be the standard Gaussian random variable. Then $X = \mu + \sigma Z$ and $Y = f(Z) = e^\mu e^{\sigma Z}$.

By following (2.76), we obtain

$$Y_N(Z) = e^{\mu+\sigma^2/2} \sum_{k=0}^{N} \frac{\sigma^k}{k!} H_k(Z) \tag{3.4}$$

where $Y_N(Z)$ is the generalized polynomial chaos (gPC) approximation of Y with order $N$ and $H_k(x)$ is the hermite polynomial with order k.
Hermite polynomials satisfy

$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x),\ n > 0, \tag{3.5}$$

and

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)w(x)dx = n!\,\delta_{nm},\ n > 0, \tag{3.6}$$

where

$$w(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$$

Note that the definition of $H_n(x)$ here is slightly different from the classical one used in the literature. Classical Hermite polynomials $\tilde{H}_n(x)$ are often defined by

$$\tilde{H}_{n+1}(x) = 2x\tilde{H}_n(x) - 2n\tilde{H}_{n-1}(x),\ n > 0, \tag{3.7}$$

and

$$\int_{-\infty}^{\infty} \tilde{H}_n(x)\tilde{H}_m(x)w(x)dx = 2^n n!\,\delta_{nm},\ n > 0, \tag{3.8}$$

where $w(x) = \frac{1}{\sqrt{\pi}}\, e^{-x^2/2}$. The two expressions are off by a scaling factor.

#### 3.3.2.2 PC Approximation

PC represents the random variable $V(S,t;\sigma(\omega))$ as a PC series. Once we have the series representation, the mean and standard deviation may be computed directly from the series coefficients as shown in (2.78) and (2.79).

We will use a stochastic collocation method [26, 27] to obtain the PC approximation. We begin by considering the price of the put $V(S,t;\sigma)$ or as we parametrized $\sigma$ with log-normal density we can define a function $g$ such that $\sigma = g(\xi)$ where $\xi \sim \mathcal{N}(0,1)$. In this case, we can use the hermite polynomials for the PC approximation of $V(S,t;\sigma)$ with $\sigma$ replaced by $g(\xi)$, $\xi$ is a standard normal random variable whose density is denoted by $f_\xi$.
Let $\phi_k(x)$ denote the $kth$ degree hermite polynomial, so that $\phi_k(\xi)$ is the $kth$ PC basis function. The first step in the stochastic collocation method is to expand the solution $V$ as

a $PC$ series.

$$V(S,t;g(\xi)) \approx \sum_{k=0}^{p} \alpha_k(S,t)\phi_k(\xi) \tag{3.9}$$

where $p$ denotes the maximum polynomial degree of the series approximation.
Then the solution $V$ is projected onto the $kth$ basis element to obtain the coefficient $\alpha_k(S,t)$

$$\alpha_k(S,t) = \int_{\Omega} V(S,t;g(\xi))\phi_k(\xi)d\mathbb{P}_{\xi}. \tag{3.10}$$

We write the integral in terms of the density $f_{\xi}$ which is the standard normal distribution

$$\alpha_k(S,t) = \int_{\mathbb{R}} V(S,t;g(x))\phi_k(x)f_{\xi}(x)dx \tag{3.11}$$

which can be approximated by a $Jth$ order Gauss-Hermite quadrature rule, i.e.

$$\alpha_k(S,t) \approx \sum_{j=1}^{J} V(S,t;g(x_j))\phi_k(x_j)w_j, \tag{3.12}$$

where $\{x_j\}_{j=1}^{J}$ and $\{w_j\}_{j=1}^{J}$ are respectively the roots and weights of a Hermite polynomial of order $J$.
In our case, we can compute $V(S,t;g(x_j))$ using the analytic solution for the European put. However, in general, we should solve a variant of Black-Scholes PDE with replacing $\sigma$ with $g(x_j)$. Once the coefficients $\alpha_k(S,t)$ are computed, we may then form the series Equation (3.9).

To compute the distribution of $V$ by PC approximation, we again use MC to generate a set $\{\sigma_i\}_{i=1}^{N}$, where each $\sigma_i$ is drawn from log-normal distribution, and for each $i$ the associated $V_i$ is computed. The difference here from Section 3.3.1 is that each $V(S,t;\sigma_i)$ is computed using Equation (3.9). Thus, regardless of which option one is analyzing, the cost of computing each sample point in the MC simulation is small (the evaluation of a sum).

The computation of the PC series has two contributions to the error. The first is determined by $p$ and the rate of decay of the coefficients. The second is determined by $J$ and the order of the Gauss quadrature rule used to approximate the orthogonal projection. We can quantify those errors by examining the spectrum of the approximation. Figure 3.3 shows the convergence of the coefficients for two values of $J$. We see that by $p = 10$ , the spectral coefficients $\alpha_k(S,t)$ are in the order of $10^{-5}$. From figure 3.3 we estimate that $p = 10$ and $J = 30$ are sufficient for an error in the computed result of Equation (3.9) of $10^{-5}$.

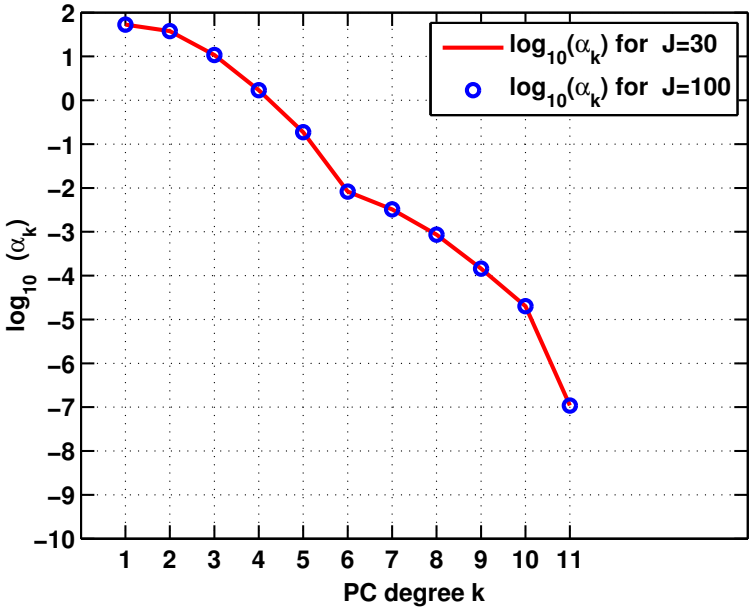
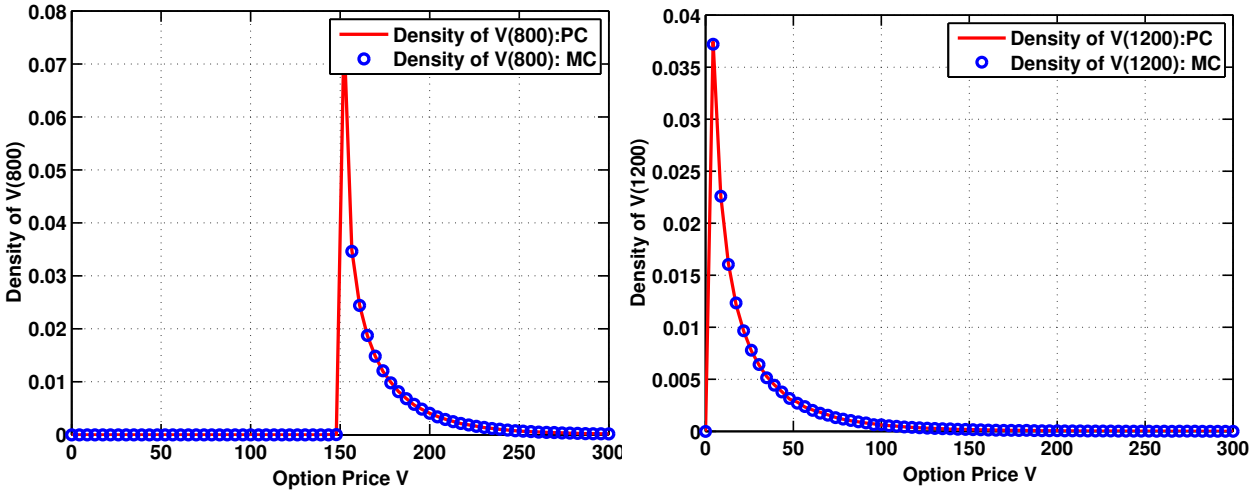



Figure 3.3: Convergence of PC coefficients

## 3.4 Numerical Results

The sample densities $V(S,t;\sigma)$ for $S = 800, 1000, 1200$ are given by Figures 3.4 and 3.5. Figure 3.4 shows that the distribution of $V(S,t;\sigma)$ is highly skewed for either deep in or out of the money options.

For near the money options, $V(S,t;\sigma)$ has a distribution that is similar (in shape) to the sample distribution of $\sigma$. The skewness is a consequence of the fact that the price of a put option, as a function $\sigma$, is an increasing function overall but becomes flat as $\sigma$ approaches zero. This insensitivity to $\sigma$ becomes more severe the further the $S$ is from the strike. Thus, for deep in or out of the money puts, small values of $\sigma$ all produce similar option prices and cause the density to place a large amount of mass near the minimum option value for the given $S$.
The comparison between the densities obtained by MC and PC methods shows that for $N = 100000$, $p = 10$, *and* $J = 30$ same results are produced .
Figures 3.6 and 3.7 show how the mean, the standard deviation and 1% bounds of $V(S,t;\sigma)$ vary in general with S for $t = 0$ *and* $N = 100000$ . The probabilistic bounds of $V(S,t;\sigma)$ are computed using the kernel density method. We conclude from these figures that the two methods (MC and PC) produce the same results. This is confirmed by Table 3.1 which gives the mean and standard deviation by MC and PC methods for $S = 800, 1000, 1200$ ( i.e for an in-the-money option, an at-the-money option, and an out-of-the money option). This table shows that the difference between these methods' outputs is of about $10^{-3}$.




(a) Density of V(800) by MC and PC methods (b) Density of V(1200) by MC and PC methods

Figure 3.4: Density of V(1200) and V(800) by MC and PC methods

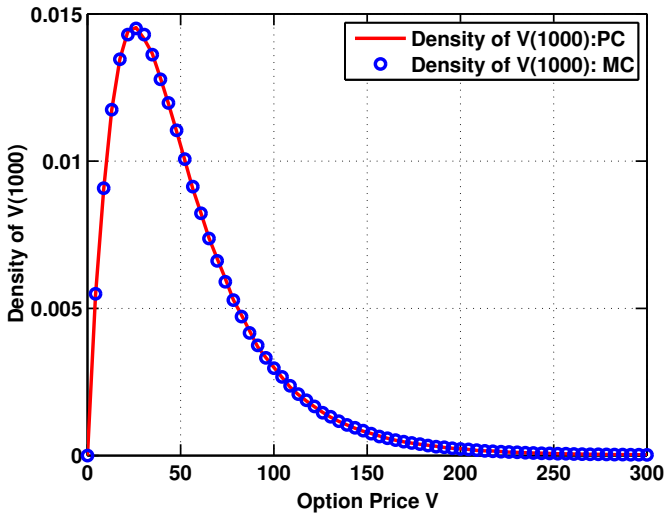
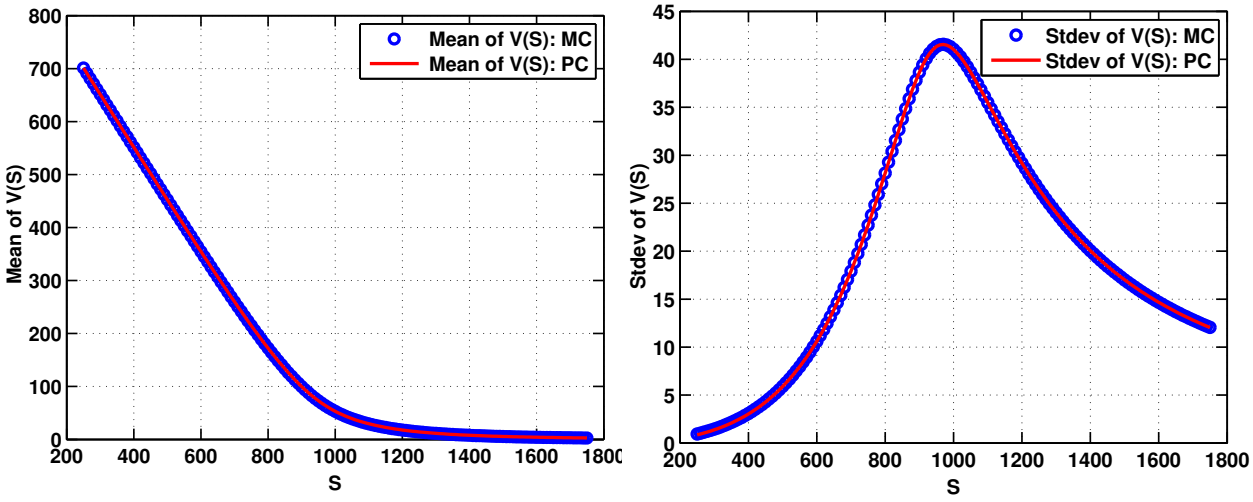



Figure 3.5: Density of V(1000) by MC and PC methods

| S | 800 | 1000 | 1200 |
|---|---|---|---|
| Mean of V(S,0,$\sigma$)(MC) | 172.1635 | 53.0375 | 17.6456 |
| Mean of V(S,0,$\sigma$)(PC) | 172.1638 | 53.0373 | 17.6454 |
| | | | |
| Stdev of V(S,0,$\sigma$)(MC) | 28.0329 | 41.0428 | 28.7341 |
| Stdev of V(S,0,$\sigma$)(PC) | 28.0328 | 41.0421 | 28.7343 |

Table 3.1: Statistics of V(S,t,$\sigma$) computed with MC and PC methods




(a) Mean of V(S) by MC and PC methods

(b) Stdev of V(S) by MC and PC methods

Figure 3.6: The mean and standard deviation of V(S, 0; $\sigma$) with $\sigma$ random

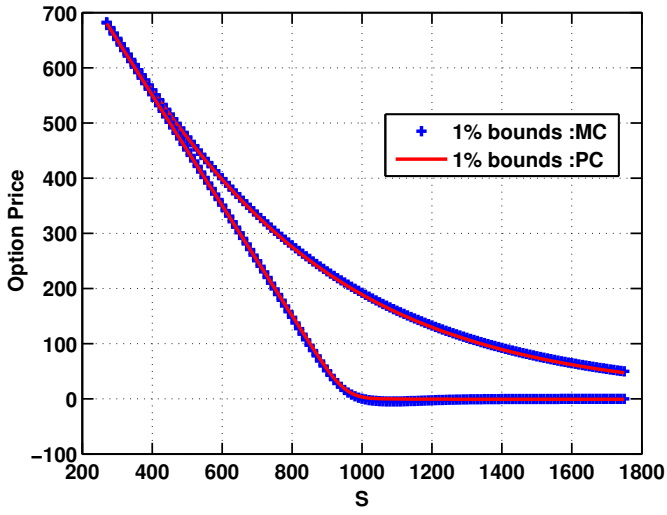



Figure 3.7: Comparison of $^{\pm}1\%$ bounds of option price V using PC and MC methods

## 3.5 Performance of Crude MC Versus Surrogate PC/MC Approximation

We now compare the efficiency of a crude MC implementation of probabilistic uncertainty and the surrogate PC/MC approximation. In both implementations, we must obtain the set of $\{V_i(S,t)\}_{i=1}^{N}$. When using crude MC, we computed the values of $V_i(S,t)$ directly, and thus $N$ applications of the option price solver are required. In the surrogate implementation, $V(S,t;\sigma)$ is approximated by a PC series. To get the PC approximation, one needs to solve the option price $J$ times. The $V_i$'s are then computed using the PC series approximation. The important point is that $J \ll N$. For the results presented, recall that $N$ was 100000

and $J$ was 30.

The computation time of a MC implementation is

$$C_{MC} = N.T_{solver}$$

and the time for the surrogate implementation is

$$C_{sur} = J.T_{solver} + N.T_{PCsum}$$

where $T_{solver}$ denotes the computation time of the option solver used and $T_{PCsum}$ denotes the time required to evaluate the PC series. Taking the ratio of the two, one obtains

$$\frac{C_{sur}}{C_{MC}} = \frac{J.T_{solver} + N.T_{PCsum}}{N.T_{solver}} = \frac{J}{N} + \frac{T_{PCsum}}{T_{solver}} \tag{3.13}$$

Equation(3.13) has two terms. The first, $\frac{J}{N}$ , is implementation independent and represents the maximal efficiency of the surrogate method. For $J = 30$ and $N = 100000$ , $\frac{J}{N} = 0.0003$ so that the surrogate method would be 3333 times faster than the crude MC for these parameters. The second term $\frac{T_{PCsum}}{T_{solver}}$, however, depends on the option solver one uses. The larger $T_{solver}$ is, the greater the benefit of using the surrogate method.

Table 3.2 gives an idea about the efficiency of MC versus PC in our case. The benefit of the surrogate method is marginal for the European option because of the availability of an inexpensive exact solution. However, if one analyses an option whose solution (analytic or numerical) requires non-trivial computation time, the benefit of the surrogate method can be significant.

| Method | Time for computing Mean of V(S) | Time for computing Stdev of V(S) |
|---|---|---|
| MC with $N = 100000$ | 0.0571 | 0.0396 |
| PC with $p = 10, J = 30$ | 0.0011 | 0.0069 |

Table 3.2: Efficiency of MC versus PC

## 3.6 Conclusion

In this chapter, we investigated the impact of volatility's uncertainty on the price of European put under Black-Scholes framework. We modeled the volatility from historical data and its density was fitted by the best parametric distribution which was in our case the log-normal

distribution. We used the probabilistic approach and implemented MC and PC methods that used this distribution to compute mean quantities, $\alpha$ bounds and sample densities of the put's price. Both methods gave same results but the surrogate PC/MC method was significantly faster. For the polynomial orders and the number of samples needed in our case, PC method was up to 3300 times faster depending on the solver's speed. Unlike deterministic methods basically Worst case analysis, probabilistic methods that we implemented here can be applied with slight modification to any kind of option or pricing model. We also mention that our work can be improved if we consider the calibrated sample density of the volatility. In this case, the orthogonal polynomials must be constructed numerically.

# Chapter 4

# UQ In Option Pricing: Multidimensional Case

## 4.1 Introduction

The Black-Scholes option pricing model in high dimensional case has two sources of uncertainty, the covariance matrix $\Sigma$ and risk free rate $r$, which must be specified. Uncertainty in either leads to uncertainty in the option price. We consider that $\Sigma$ is the main source of uncertainty given the relatively short time span of the option. In this case, our aim is to investigate the effect of uncertainty in the covariance matrix $\Sigma$ on option pricing by computing the mean of the basket option price as well as estimating its sample density.

After analyzing different approximation methods of arithmetic basket option price and selecting the best one, we develop three ways in order to compute the mean of basket option price:

- Nested MC Simulation.
- MC + Approximation (MC/Approx).
- SGQ +Approximation.

We start this chapter with parameterizing the uncertainty in the covariance matrix $\Sigma$ and analyzing pricing approximations for basket options in order to select the best one. In the fourth section, we are going to explain the implementation of different methods used for the computation of basket option price mean. Before concluding, we are going to present the obtained results which includes the mean and the sample density of basket put price joined with a comparison of implemented methods' performance. All numerical tests are performed with Matlab.

## 4.2 Uncertainty Parametrization

To model the randomness in the covariance matrix of asset's returns, we use the Wishart distribution. In fact, the Wishart distribution is often used to model random covariance matrices(see [35–37]).

A Wishart random matrix with parameters $n$ and $H$ can be seen as a sum of outer products of $n$ independent multivariate normal random vectors having mean 0 and covariance matrix $\frac{1}{n}H$ . It is characterized as follows:

**Definition 4.2.1.** *Let $W$ be a $K \times K$ absolutely continuous random matrix. Let its support be the set of all symmetric and positive definite real matrices:*

$$R_w = \{w \in \mathbb{R}^{K \times K} : w \, is \, symmetric \, and \, positive \, definite\}$$

*Let $H$ be a symmetric and positive definite matrix and $n > K - 1$. We say that $W$ has a Wishart distribution with parameters $n$ and $H$ if its joint probability density function is:*

$$f_W(w) = c[det(w)]^{n/2-(K+1)/2} exp(-\frac{n}{2} tr(H^{-1}w)) \tag{4.1}$$

*where*

$$c = \frac{n^{n/2}}{2^{nK/2}[det(H)]^{n/2}\pi^{K(K-1)/4}\prod_{j=1}^{K}\Gamma(n/2 + (1-j)/2)}$$

*and $\Gamma$ is the Gamma function.*

The parameter $n$ needs not be an integer, but, when $n$ is not an integer, $W$ can no longer be interpreted as a sum of outer products of multivariate normal random vectors.
The following proposition provides the link between the multivariate normal distribution and the Wishart distribution:

**Proposition 4.2.1.** *Let $X_1, ..., X_n$ be $n$ independent $K \times 1$ random vectors all having a multivariate normal distribution with mean 0 and covariance matrix $\frac{1}{n}H$. Let $K = n$. Define:*

$$W = \sum_{i=1}^{n} X_i X_i^T$$

*Then $W$ has a Wishart distribution with parameters $n$ and $H$.*

The expected value of a Wishart random matrix $W$ with parameters $n$ and $H$ is:

$$E[W] = H$$

.

## 4.3 Comparison of Approximation Methods For Basket Option Pricing

This section analyses the performance of the particular pricing approximation methods mentioned in 2.1.5 which are ( Ju, Levy and Beisser approximations). We will do this by changing all relevant parameters individually, while the remaining ones are kept fixed to our standard scenario.

That means in detail that four sets of tests are performed. They involve changing the correlations $\rho_{ij}$, strike $K$, initial stock prices $S_i^0$ and the volatilities $\sigma_i$ respectively.
As a standard scenario we use a basket call composed by 4 stocks and having five years as maturity. The discount factor is equal to one. Let $i$ and $j$ denote the indices of the stocks. The default parameters are:

$$T = 5\ ;\ r = 0\ ;\ \rho_{ij} = 0.5(i \neq j)\ ;\ K = 100\ ;\ S_i(0) = 100\ ;\ \sigma_i = 0.4\ ;\ \omega_i = \frac{1}{4}$$

In the following, we are going to compare the results of different approximations to Monte Carlo with Control Variate technique.
The deviation to Monte Carlo denoted by 'Dev' is used as a performance index for the different methods and calculated as

$$Dev = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(MC_i - V_i)^2}, \tag{4.2}$$

where $MC_i$ is the basket option price calculated by Monte Carlo method and $V_i$ is the price calculated using one of pricing approximations ( Ju, Levy or Beisser approximations).

### 4.3.1 Varying The Correlations

Figure 4.1 and table 8 in appendix A show the effect of simultaneously changing all correlations from $\rho = \rho_{ij} = 0,1$ to 0.95. All methods perform reasonably well, especially for $\rho \geq 0.8$, all methods give the same price except Beisser approximation.

We can explain the good performance of Ju and Levy for high correlations as follows: these methods provide exactly the Black-Scholes prices for the special case that the number of stocks is one. For high correlations the distribution of the basket is approximately the sum of the same( for $\rho = 1$ exactly the same) log-normal distributions, which is indeed again log-normal.

In total the prices calculated by Ju's approach is the closest to the Monte Carlo prices.

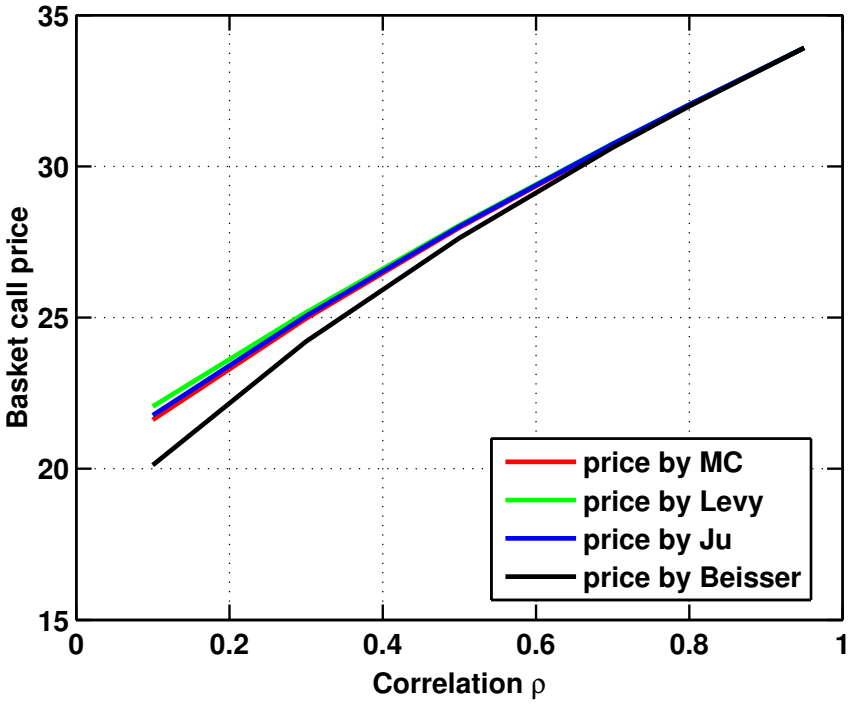



Figure 4.1: Varying correlation

### 4.3.2 Varying The Strike

With all other parameters set to the default values, the strike $K$ is varied from 50 to 150. Figure 4.2 and table 9 in appendix A show the results. Again, overall Ju's approximation performs best. The relative and absolute differences of all methods are generally increasing when K is growing, since the approximation of the real distributions in the tails is getting worse and the absolute prices are decreasing.

### 4.3.3 Varying Stock Prices

The prices of all stocks are now set to the same value $S^0$ which is varied between 50 and 150 in this set of tests. Figure 4.3 and table 10 in appendix A show that Ju's method is the best approximation compared to Monte Carlo .

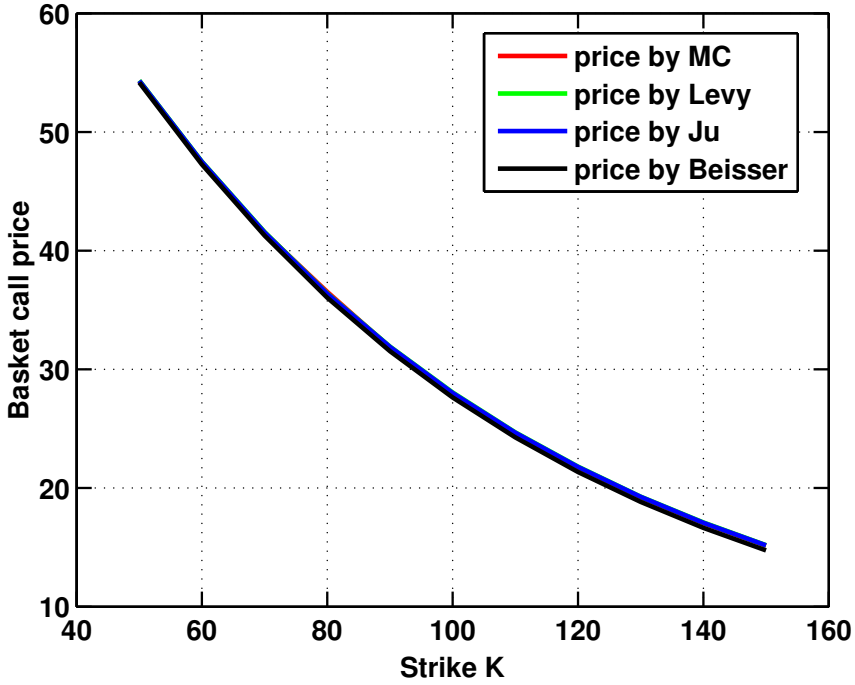



Figure 4.2: Varying strike

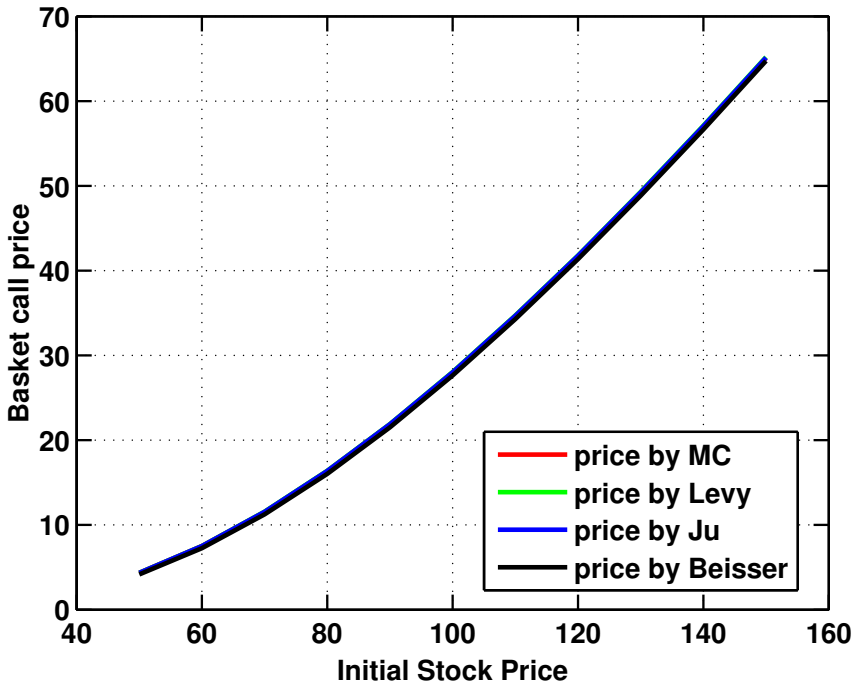



Figure 4.3: Varying stock price

### 4.3.4 Varying The Volatilities

We start with the symmetrical situation at each step, $\sigma_i$ is set to the same value $\sigma$, which is varied between 5% and 100%. Figure 4.4 and table 11 in appendix A show the results of

the test. The prices calculated by the different methods are more or less equal for "small" values of the volatility. They start to diverge at $\sigma \approx 60\%$ . And again Ju's approximation is the best.

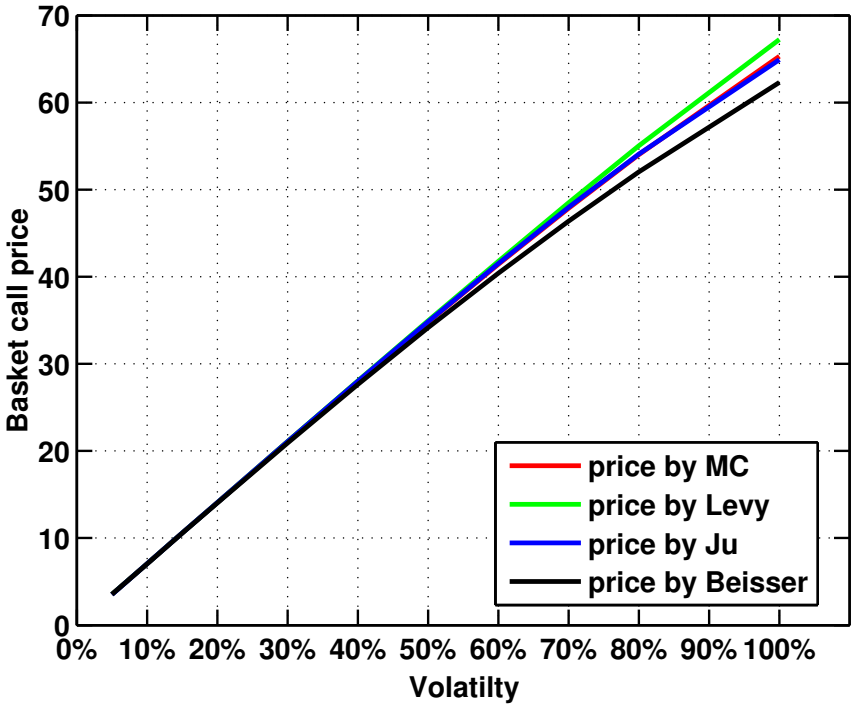



Figure 4.4: Varying volatilty symetrically

| Volatility | Beisser | Ju | Levy | Monte Carlo CV | StdDev |
|---|---|---|---|---|---|
| 5% | 19,45 | 35,59 | 55,56 | 22,65 | (0,55) |
| 10% | 20,84 | 36,19 | 55,52 | 21,3 | (0,38) |
| 15% | 22,6 | 36,93 | 55,61 | 22,94 | (0,26) |
| 20% | 24,69 | 37,8 | 55,71 | 25,24 | (0,21) |
| 30% | 29,52 | 39,97 | 55,98 | 30,95 | (0,16) |
| 40% | 34,72 | 42,66 | 56,35 | 36,89 | (0,11) |
| 50% | 39,96 | 45,84 | 56,89 | 41,72 | (0,08) |
| 60% | 45,05 | 49,39 | 57,68 | 46,68 | (0,04) |
| 70% | 49,88 | 53,21 | 58,87 | 51,78 | (0,05) |
| 80% | 54,39 | 57,17 | 60,70 | 56,61 | (0,07) |
| 100% | 62,32 | 64,93 | 67,24 | 65,31 | (0,09) |
| Dev | 1,92 | 8,96 | 22,70 | | |

Table 4.1: Varying the volatilities asymmetrically

Ju's approximation is no longer the best when we have asymmetry in the volatilities, precisely if there are groups of stocks with high and with low volatilities entering the basket. This is clearly demonstrated by Table 4.1, where we fix $\sigma_1 = 100\%$ and vary the remaining

volatilities symmetrically. We note also the extremely bad performance for Levy's method for small values of $\sigma$. Beisser seems to be the best approximation in case of inhomogeneous volatilities.

### 4.3.5 Conclusion

As it was shown by different tests, the Ju's method is the best approximation except for the case of inhomogeneous volatilities. The reason for this drawback may be that all stocks are "thrown" together on one distribution. This is quite contrary to Beisser's approximation, where every single stock keeps a transformed log-normal distribution and the expected value of every stock is individually evaluated. This is probably the reason why this method is able to handle the case of inhomogeneous volatilities.

Beisser's approximation underprices slightly in all cases since this method is a lower bound on the true option price. Beisser's approach is the only method which is reliable in the case of inhomogeneous volatilities.

Giving these results, in the following, we will use Ju and Beisser methods to investigate the effect of parametric uncertainty in multidimensional case.

## 4.4 Implementation Details

### 4.4.1 Introduction

Consider a basket of n assets $S_1, S_2, .....S_d$ let $r$ denote the risk-free short rate, and suppose that the risk-neutralized price process $S_i(t)$ satisfies

$$dS_i(t) = S_i(t)(rdt + \sigma_i dW_i(t)) \tag{4.3}$$

where $\sigma_i$ is the volatility of the asset $i$, $W_i(t)$ is a standard Brownian motion, and

$$\mathrm{E}[dW_i(t), dW_j(t)] = \rho_{ij}dt. \tag{4.4}$$

Let $V(S_1, S_2, .....S_d, \Sigma)$ denote the price of a basket option of $d$ assets and which depends on the covariance matrix of these assets $\Sigma$. If we suppose that the covariance matrix $\Sigma$ follows Wishart distribution denoted by $\rho(\Sigma)$ and we define $\Pi(\Sigma)$ one of the approximation functions that we selected for the valuation of the basket option price then

$$\mathrm{E}(V(S_1, S_2, .....S_d, \Sigma)) = \int_{\mathbb{R}^{\frac{d\times(d+1)}{2}}} \Pi(\Sigma)\rho(\Sigma)d\Sigma. \tag{4.5}$$

Based on (4.5) we computed the mean of the basket option price via 3 ways :

- Nested MC Simulation
- MC + Approximation (MC/Approx)
- SGQ +Approximation

In the following we will explain the implementation of each method.

### 4.4.2 Mean Computation Of Basket Put Price Using Nested MC Simulation

In this section, we compute the mean of basket put option price using Nested MC Simulation applied to the left side (4.5) which means without using an approximation for the price.

Giving a covariance matrix $\Sigma$, we define $R$ a $d \times d$ correlation matrix with entries $R_{ij} = \rho_{ij}$ and let L be the solution of $LL^T = R$ obtained by the Cholesky factorization (see [16] p.73 for an algorithm to compute L).Then we get the following form used for the simulation

$$S_i(T) = S_i(0)exp\left(\left(r - \sigma_i^2/2\right)T + \sigma_i\sqrt{T}\sum_{j=1}^{i} L_{ij}\xi_j\right), i = 1, ...., d \tag{4.6}$$

where $\xi_j$ , $j = 1, ....., d$ are independent standard normal random variates. Note that the $i-th$ element of the vector $L\xi$ can be written as $\sum_{j=1}^{i} L_{ij}\xi_j$ as $L$ is lower triangular. Algorithm 2 shows the implementation details.

One way to reduce the variance of MC methods is to combine it with Control variate technique. If we consider an option that uses geometric averages as the geometric average of log-normal variate is itself log-normal. A geometric average call has the payoff function $P_G = (K - G)^+$ where $G = exp(\sum_{i=1}^{d} w_i \ln S_i)$. The payoffs of the geometric and the arithmetic average options are very close to each other as the values of $S_i$ are close. In fact, arithmetic and geometric averages yield the same result if all prices are the same. So, it is a sensible choice to use the payoff of the geometric average option $P_G$ as a control variate (CV) of $P_A$ with $P_A$ is the arithmetic average call. The simulation estimator for the price (without the discount factor $e^{-rT}$) is $Y_{CV} = P_A - c(P_G - \mu_{P_G})$, where

$$\mu_{P_G} := \mathrm{E}[P_G] = e^{\mu_{\tilde{s}} + \sigma_{\tilde{s}}^2/2}\Phi(-k + \sigma_{\tilde{s}}) - K\Phi(-k), \tag{4.7}$$

where $\Phi$ denotes the cumulative distribution function (cdf) of the standard normal distribution,

$$k = \frac{\ln K - \mu_{\tilde{s}}}{\sigma_{\tilde{s}}}, \tag{4.8}$$

and $\mu_{\tilde{s}} = \mathrm{E}[\ln G]$ , $\sigma_{\tilde{s}}^2 = \mathrm{Var}(\ln G)$, which are given by equations $\mu_{\tilde{s}} = \sum_{i=1}^{d} \omega_i \tilde{\mu}_i$ and

**Algorithm 2** MC for mean computation of basket put option price

**Inputs** Sample size $n_1$ for random covariance matrices, Sample size $n_2$ for asset prices generation, maturity $T$, time series length of assets' returns $p$, number of assets $d$, weights of assets $\omega_i$, initial asset prices $S_i(0)$, Strike price K, covariance matrix $R$, risk free interest rate $r$, Confidence Coefficient $\alpha$.

**for** $m = 1.....n_1$ **do**

Generate a random covariance matrix $\Sigma$ following Wishart distribution with parameters $R$ and $p$.

Compute the new volatilities vector $\sigma$ and correlation matrix $\rho$ from $\Sigma$

Compute the Cholesky factor $L$ of $\rho$.

**for** $i = 1.....n_2$ **do**

Generate independent standard normal variates, $\xi \sim \mathcal{N}(0,1)$, $j = 1, ....., d$

Set $S_j(T) \leftarrow S_j(0)exp\left(\left(r - \sigma_j^2/2\right)T + \sigma_j\sqrt{T}\sum_{k=1}^{j} L_{jk}\xi_k\right), j = 1, ...., d.$

Set $Y_i \leftarrow e^{-rT}\left(K - \sum_{j=1}^{d} w_j S_j(T)\right)^+$

**end for**

Compute the sample mean $Y_m$ of $Y_i$

**end for**

Compute the sample mean $\bar{P}$ and the sample standard deviation $s$ of $Y_m$.

**return** $\bar{P}$ and the error bound $\Phi^{-1}(1-\alpha/2)s/\sqrt{n_1}$,where $\Phi^{-1}$ denotes the quantile of the standard normal distribution.

$\sigma_{\tilde{s}}^2 = \sum_{i=1}^{d}\sum_{j=1}^{d}\omega_i\omega_j\tilde{\sigma}_i\tilde{\sigma}_j\tilde{\rho}_{ij}$ where

$$\tilde{\mu}_i = \mathrm{E}[\ln S_i] = \ln S_i(0) + (r - \sigma_i^2/2)T \tag{4.9}$$

$$\tilde{\sigma}_i = \sqrt{\mathrm{Var}(\ln S_i)} = \sigma_i\sqrt{T} \tag{4.10}$$

$\tilde{\rho}_{ij}$ is the correlation between $\ln S_i$ and $\ln S_j$ and we have $\tilde{\rho}_{ij} = \rho_{ij}$. The CV coefficient $c$ is considered as an input to our algorithm. Since the optimal $c^* = \mathrm{Cov}(P_A, P_G)/\mathrm{Var}(P_G)$ is very close to one in most cases, we can choose simply $c = 1$.

The details of MC method using Control variate to compute the mean of basket put option price are shown by algorithm 3.

**Algorithm 3** MC using Control variate technique for mean computation of basket put option price

**Inputs** Sample size $n_1$ for random covariance matrices realizations, Sample size $n_2$ for asset prices generation, time series length of assets' returns $p$, maturity $T$, number of assets $d$, weights of assets $\omega_i$, initial asset prices $S_i(0)$, Strike price K, covariance matrix $R$, risk free interest rate $r$, confidence Coefficient $\alpha$, CV coefficient c(can be set to 1).

**for** $i = 1.....n_1$ **do**

Generate a random covariance matrix $\Sigma$ following Wishart distribution with parameters $R$ and $p$.

Compute the new volatilities vector $\sigma$ and correlation matrix $\rho$ from $\Sigma$

Compute the Cholesky factor $L$ of $\rho$.

Compute $\mu_P$ by using Equation 4.7

**for** $i = 1.....n_2$ **do**

Generate independent standard normal variates, $\xi \sim \mathcal{N}(0,1)$, $j = 1, ....., d$

Set $S_j(T) \leftarrow S_j(0)exp\left(\left(r - \sigma_j^2/2\right)T + \sigma_j\sqrt{T}\sum_{k=1}^{j} L_{jk}\xi_k\right), j = 1, ...., d.$

Set $P_A \leftarrow \left(\sum_{j=1}^{d} \omega_j S_j(T) - K\right)^+$.

Set $P_G \leftarrow \left(exp\left(\sum_{j=1}^{d} \omega_j \ln S_j(T)\right) - K\right)^+$.

Set $Y_i \leftarrow e^{-rT}\left(P_A - c\left(P_G - \mu_{P_G}\right)\right) + \left(e^{-rT}K - \sum_{j=1}^{d} \omega_j S_j(0)\right)$

**end for**

Compute the sample mean $Y_m$ of $Y_i$

**end for**

Compute the sample mean $\bar{P}$ and the sample standard deviation $s$ of $Y_m$.

**return** $\bar{P}$ and the error bound $\Phi^{-1}(1-\alpha/2)s/\sqrt{n_1}$,where $\Phi^{-1}$ denotes the quantile of the standard normal distribution.

### 4.4.3 Mean Computation Of Basket Put Price Using MC With Approximation

In this section, we compute the mean of basket put option price with MC applied to(4.5) which means after using an approximation method for the price. As it was concluded in 4.3, we will use both Beisser and Ju approximation methods as they are the best. The details of our implementation are presented by algorithm 4.

**Algorithm 4** Mean computation of basket put option price using MC/Approx

**Inputs** Sample size $n$ for random covariance matrices realizations, maturity $T$, time series length of assets' returns $p$, number of assets $d$, weights of assets $\omega_i$, initial asset prices $S_i(0)$, Strike price K, covariance matrix $R$, risk free interest rate $r$, Confidence Coefficient $\alpha$.

**for** $i = 1.....n$ **do**
Generate a random covariance matrix $\Sigma$ following Wishart distribution with parameters $R$ and $p$.
Compute $Y_i$ the approximation of the price with Beisser or Ju method using $\Sigma$ as an input parameter.
**end for**

Compute the sample mean $\bar{P}$ and the sample standard deviation $s$ of $Y_i$.

**return** $\bar{P}$ and the error bound $\Phi^{-1}(1-\alpha/2)s/\sqrt{n}$,where $\Phi^{-1}$ denotes the quantile of the standard normal distribution.

### 4.4.4 Mean Computation Of Basket Put Price Using Sparse Grid Quadrature

In this section, we compute the mean of basket put option price with SGQ applied to equation(4.5) which means after using an approximation method for the price. As it was explained in 2.3, this method can be a convenient technique for an efficient numerical treatment of multivariate problems specially in the case of multidimensional integrals.

One constraint that can deteriorates the performance of SGQ is the high dimension of the problem. In fact, in our case, if we suppose that the covariance matrix $\Sigma$ is a $d \times d$ real symmetric matrix then the dimension of the problem will be $\frac{d\times(d+1)}{2}$. So, the first step is the reduction of problem dimension which we are going to deal with in the following section.

#### 4.4.4.1 Dimension Reduction Procedure

As the covariance matrix $\Sigma$ is a real symmetric matrix then, there exists an orthogonal matrix P such that

$$D = P^{-1}\Sigma P \tag{4.11}$$

where $P^{-1} = P^{tr}$ and $D$ is a diagonal matrix with diagonal entries being the eigenvalues of $\Sigma$.

Giving this property, we can reduce the dimension of the problem to $d$ by only perturbing the eigenvalues of $\Sigma$ and keeping its eigenvectors fixed ( for more details about eigenvalues perturbation technique see [38, 39])).

Let $(\lambda_1, \lambda_2, ....., \lambda_d)$ be the eigenvalues of $\Sigma$ then equation (4.5) becomes

$$\begin{aligned} \mathrm{E}(V(S_1, S_2, .....S_n, \Sigma)) &= \int_{\mathbb{R}^{\frac{d\times(d+1)}{2}}} \Pi(\Sigma)\rho(\Sigma)d\Sigma \\ &= \int_{\mathbb{R}^d} \Pi(\lambda_1, \lambda_2, ....., \lambda_d)\rho_{\lambda_1,\lambda_2,.....,\lambda_d}(\lambda_1, \lambda_2, ....., \lambda_d)d\lambda_1 d\lambda_2 ...d\lambda_d. \end{aligned}$$

We follow these steps in order to perturb the eigenvalues and reduce the dimension of the problem:

1. Sampling $n$ random covariance matrices following Wishart distribution with parameters $H$ and $p$.($H$ is the Covariance matrix obtained from data and $p$ is the size of time series data).
2. From these samples we recover $n$ samples for each eigenvalue by diagonalization.
3. Fitting the distribution of each eigenvalue.

Once we have the parameters of eigenvalues distributions, we can apply SGQ.

**Comment 4.4.1.** *The results of fitting eigenvalues densities( see appendices B and C)show that for all eigenvalues, the normal distribution with different parameters best fits these densities. Therefore, we are going to use Gauss hermite quadrature as the univariate quadrature in SG method. We know that this will introduce negative eigenvalues with finite probability but for these computations this event happens with very low probability and it is disregarded.*

#### 4.4.4.2 Implementation of Sparse Grid Quadrature

The implementation of SGQ is shown by algorithm 7. Note that $\Delta_k f$ is a d-dimensional product quadrature rule that can be computed by algorithm 5. To find the indices $k \in \mathbb{N}^d$ with $\mid k \mid_1 = l$ we use algorithm 8.

**Algorithm 5** Computation of the product quadrature rule

$If = 0$ , $p = 1$ and $i = (1, ..., 1)$
**repeat**
  $If = If + w_{i_1,1}.....w_{i_d,d} f(x_{i_1,1}, ....., x_{i_d,d})$
  find the next index $i = (i_1, ...., i_d)$ by algorithm 6  /⋆ input: $p$ and $i$ ⋆/
**until** $i = 0$
**return** $P_n f = If$

**Algorithm 6** Drop algorithm for the iterative enumeration of all product indices $i = (i_1, ...., i_d) \in \mathbb{N}^d$ with $i_j \in \{1, ....., n_j\}$.

**inputs:** $p \in \mathbb{N}$ *and* $i \in \mathbb{N}^d$
**repeat**
  $i_p = i_p + 1$
  **if** $i_p > n_p$ **then**
    **if** $p = d$ **then**
      **return** 0  /⋆ all indices already enumerated ⋆ /
    **end if**
    $i_p = 1$
    $p = p + 1$
  **else**
    $p = 1$
    **return** $i$  /⋆ the next index to use ⋆ /
  **end if**
**until** $i = 0$

**Algorithm 7** Implementation of the sparse grid method

$If = 0$ *and* $m = l + d - 1$
**for** $l = d, ...., m$ **do**
  $p = 1$ , $k = (m, 1, ..., 1)$ *and* $\hat{k} = (m, ..., m)$
  **repeat**
    compute the the product formula $\Delta_k f$ by algorithm 5
    $If = If + \Delta_k f$
    find the next index $k$ with $|k|_1 = l$ by algorithm 8  /⋆ input: $p$ , $k$ and $\hat{k}$ ⋆ /
  **until** $k = 0$
**end for**
**return** $SG_l f = If$

**Algorithm 8** Drop algorithm for the iterative enumeration of all indices $k \in \mathbb{N}^d$ , $k_j > 0$ *with* $\mid k \mid_1 = l$

**Inputs:** $p \in \mathbb{N}$ *and* $k$ , $\hat{k} \in \mathbb{N}^d$
**repeat**
  $k_p = k_p + 1$
  **if** $k_p > \hat{k}_p$ **then**
    **if** $p = d$ **then**
      **return** 0 /⋆ all indices already enumerated ⋆ /
    **end if**
    $k_p = 1$
    $p = p + 1$
  **else**
    **for** $j = 1, 2, ....., p-1$ **do**
      $\hat{k}_j = \hat{k}_p - k_p + 1$
    **end for**
    $k_l = \hat{k}_l$
    $p = 1$
    **return** $k$ /⋆ the next index to use ⋆ /
  **end if**
**until** $k = 0$

## 4.5 Numerical Results

### 4.5.1 Introduction

In this section, we are going to illustrate the main numerical results obtained by applying the the different methods that we implemented in 4.4. In fact, we examine two different cases: Basket put option with 4 assets and another with 6 assets.

For both problems, we take our data from the DAX 30 index listings in 2012. Variables with values common to both problems are:

- $S_i(0) = K$ , so that initially the assets are the strike price.
- the strike $K = 1$.
- the expiry $T = 1$ year.
- the risk neutral rate $r = 0.05$.

The parameter $n_2$ used in Nested MC Simulation( see algorithm 2 and 3) for the valuation of the price is equal to $10^5$ for both of the problems.

### 4.5.2 The Case of Four Dimensional Basket Put Option

The data used for the four dimensional basket put is given in table 4.2.

| Asset $S_i$ | $\omega_i$ | $\sigma_i$ | $\rho_{ij}$ | | | |
|---|---|---|---|---|---|---|
| DBK | 0.25 | 0.3978 | 1.000 | 0.752 | 0.782 | 0.659 |
| CBK | 0.25 | 0.5030 | 0.752 | 1.000 | 0.682 | 0.574 |
| ALV | 0.25 | 0.2407 | 0.782 | 0.682 | 1.000 | 0.755 |
| MUV2 | 0.25 | 0.2173 | 0.659 | 0.574 | 0.755 | 1.000 |

Table 4.2: Four dimensional basket put data

We initially look at the results obtained by using SGQ (Table 4.3 and graph 4.5 ). We see that by level 10 the mean of basket put option price converges to 0.0930 with a difference of about $10^{-4}$ with the following level. The number of nodes used in SGQ to reach such difference is 23023.

| Level(l) | Mean of the Price | number of nodes | CPU time | $log_{10}(\mid SGQ_{l+1} - SGQ_l \mid)$ |
|---|---|---|---|---|
| 2 | 0.0497 | 8 | 0.13 | $-1.7110$ |
| 3 | 0.0691 | 45 | 0.56 | $-1.9303$ |
| 4 | 0.0809 | 165 | 2.06 | $-2.2003$ |
| 5 | 0.0872 | 494 | 6.07 | $-2.5039$ |
| 6 | 0.0903 | 1278 | 16.50 | $-2.8314$ |
| 7 | 0.0918 | 2958 | 46.00 | $-3.1769$ |
| 8 | 0.0925 | 6270 | 84.53 | $-3.5365$ |
| 9 | 0.0927 | 12375 | 174.80 | $-3.9073$ |
| 10 | 0.0930 | 23023 | 444.22 | $-4.2873$ |

Table 4.3: Numerical results of Sparse Grid Quadrature : Four Dimensional Basket Put

| Method | Price Mean | Std error | CI (95%) | CPU time |
|---|---|---|---|---|
| Nested MC ($n_1 = 10^3$ ) | 0.0932 | $2.90\,e-003$ | [0.087 0.098] | $0.85\,e+003$ |
| Nested MC ($n_1 = 10^4$ ) | 0.0931 | $9.36\,e-004$ | [0.091 0.095] | $9.77\,e+003$ |
| Nested MC+CV ($n_1 = 10^3$ ) | 0.0934 | $3.02\,e-004$ | [0.092 0.094] | $2.68\,e+003$ |
| Nested MC+CV ($n_1 = 10^4$ ) | 0.0931 | $8.73\,e-005$ | [0.093 0.093] | $2.37\,e+004$ |
| Beisser's Approx+ MC ($n = 10^5$ ) | 0.0931 | $1.55\,e-005$ | [0.093 0.093] | $1.30\,e+003$ |
| Ju's Approx + MC ($n = 10^5$ ) | 0.0931 | $1.55\,e-005$ | [0.093 0.093] | $1.29\,e+003$ |
| SG (level10, 23023 pts) | 0.0930 | | | 444.22 |

Table 4.4: The mean of basket option price by different methods: Case of four assets

Table 4.4 illustrates the performance of different methods used to compute the mean of basket put price. We see that the different methods converge to the same value with a difference

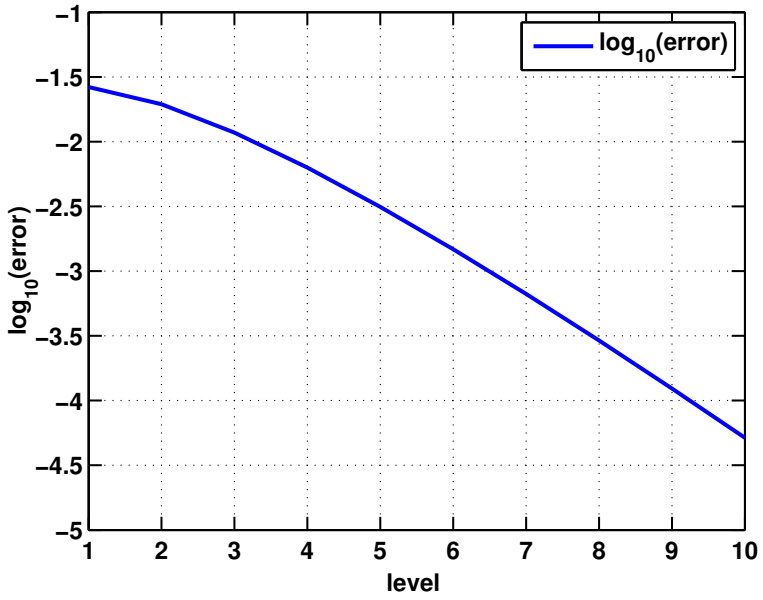



Figure 4.5: Error of Sparse Grid Quadrature($log_{10}(\mid SGQ_{l+1} - SGQ_l \mid)$): Four Dimensional Basket Put Price

of about $10^{-4}$. As we explained before, using nested MC simulation to evaluate the mean of the basket put price is very time consuming so that using approximation methods with MC or SGQ is more efficient for the computation of the basket put price mean.
SGQ was the best performed method in this case. This is can be explained by the fact that it needs less nodes ( 23023 pts) than MC/Approx($10^5$ pts) to converge to the same value with a difference of about $10^{-4}$.

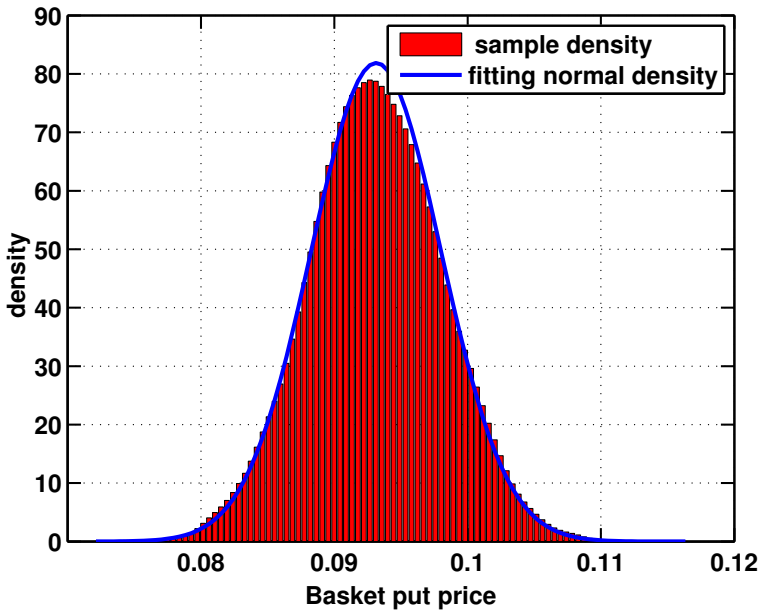



Figure 4.6: Density of four dimensional basket put price

Figure 4.6 shows the distribution of the basket put price. This was computed using( MC+Ju approximation ) and the kernel density estimation. This sample density is close to Gaussian. In fact, a Gaussian distribution with parameters $(0.0931, 0.0048)$ models the sample of basket put price reasonably.

### 4.5.3 The Case of Six Dimensional Basket Put Option

The data used for the six dimensional basket put is given in table 4.5.

| Asset $S_i$ | $\omega_i$ | $\sigma_i$ | $\rho_{ij}$ | | | | | |
|---|---|---|---|---|---|---|---|---|
| DBK | 1/6 | 0.3978 | 1.000 | 0.752 | 0.782 | 0.659 | 0.584 | 0.586 |
| CBK | 1/6 | 0.5030 | 0.752 | 1.000 | 0.682 | 0.574 | 0.481 | 0.482 |
| ALV | 1/6 | 0.2407 | 0.782 | 0.682 | 1.000 | 0.755 | 0.631 | 0.646 |
| MUV2 | 1/6 | 0.2173 | 0.659 | 0.574 | 0.755 | 1.000 | 0.596 | 0.543 |
| DAI | 1/6 | 0.3002 | 0.584 | 0.481 | 0.631 | 0.596 | 1.000 | 0.832 |
| BMW | 1/6 | 0.3002 | 0.586 | 0.482 | 0.646 | 0.543 | 0.832 | 1.000 |

Table 4.5: Six dimensional basket put data

From the results shown on table 4.6 and graph 4.7 we see that by level 9 the mean of basket put option price converges to 0.082 with a difference of about $10^{-3.5}$ with the following level. The number of nodes used in SGQ to reach such difference is 125879.

| Level | Mean of the Price | number of nodes | CPU time | $log_{10}(\mid SGQ_{l+1} - SGQ_l \mid)$ |
|---|---|---|---|---|
| 2 | 0.0280 | 7 | 0.62 | $-1.7442$ |
| 3 | 0.0460 | 88 | 3.35 | $-1.8450$ |
| 4 | 0.0603 | 454 | 16.35 | $-2.0130$ |
| 5 | 0.0700 | 1820 | 66.70 | $-2.2218$ |
| 6 | 0.0760 | 6188 | 224.57 | $-2.4815$ |
| 7 | 0.0793 | 18563 | 740.46 | $-2.7447$ |
| 8 | 0.0811 | 50375 | $1.46\,e+003$ | $-3.0458$ |
| 9 | 0.0821 | 125879 | $2.14e+003$ | $-3.5112$ |

Table 4.6: Numerical results of Sparse Grid Quadrature : Six dimensional basket put option

Table 4.7 illustrates the performance of different methods used to compute the mean of basket put price in the case of six dimensional Basket put. We see that the different methods converge to the same value 0.082 with a difference of about $10^{-4}$. SGQ is no longer the best performed method in this case. In fact, It needs more nodes (125879 pts) than MC/Approx ($10^5$ pts) to converge.

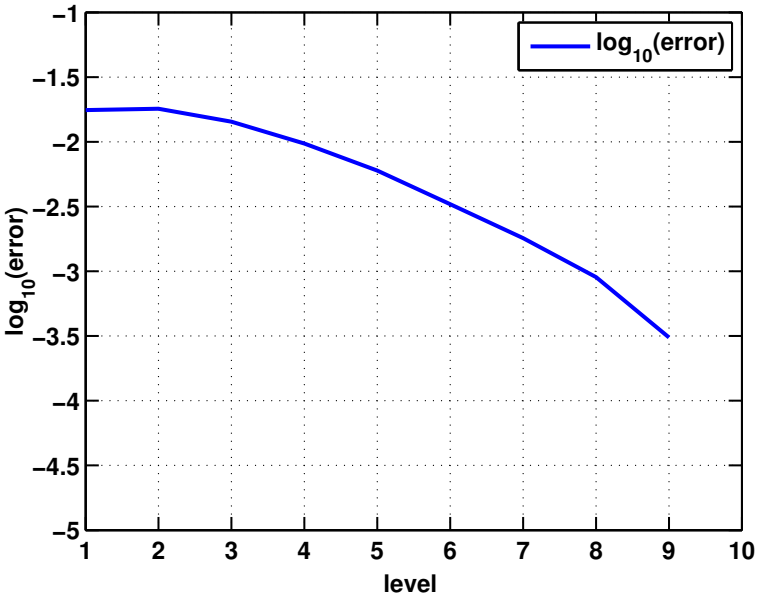



Figure 4.7: Error of Sparse Grid Quadrature($log_{10}(\mid SGQ_{l+1} - SGQ_l \mid)$): Six Dimensional Basket Put Price

| Method | Price Mean | Std error | CI(95%) | CPU time |
|---|---|---|---|---|
| Nested MC ($n_1 = 10^3$ ) | 0.0826 | $2.65\,e-003$ | [0.077 0.087] | 758.32 |
| Nested MC ($n_1 = 10^4$ ) | 0.0829 | $9.16\,e-004$ | [0.081 0.084] | $7.07\,e+003$ |
| Nested MC+CV ($n_1 = 10^3$ ) | 0.0828 | $3.15\,e-004$ | [0.082 0.083] | $2.22\,e+003$ |
| Nested MC+CV ($n_1 = 10^4$ ) | 0.0829 | $9.72\,e-005$ | [0.082 0.083] | $2.13\,e+004$ |
| Beisser's Approx + MC ($n = 10^5$ ) | 0.0827 | $1.42\,e-005$ | [0.082 0.082] | $2.05\,e+003$ |
| Ju's Approx + MC ($n = 10^5$ ) | 0.0829 | $1.41\,e-005$ | [0.082 0.082] | $1.92\,e+003$ |
| SG (level9, 125879 pts) | 0.0821 | | | $2.14e+003$ |

Table 4.7: The mean of basket option price by different methods: Case of six assets

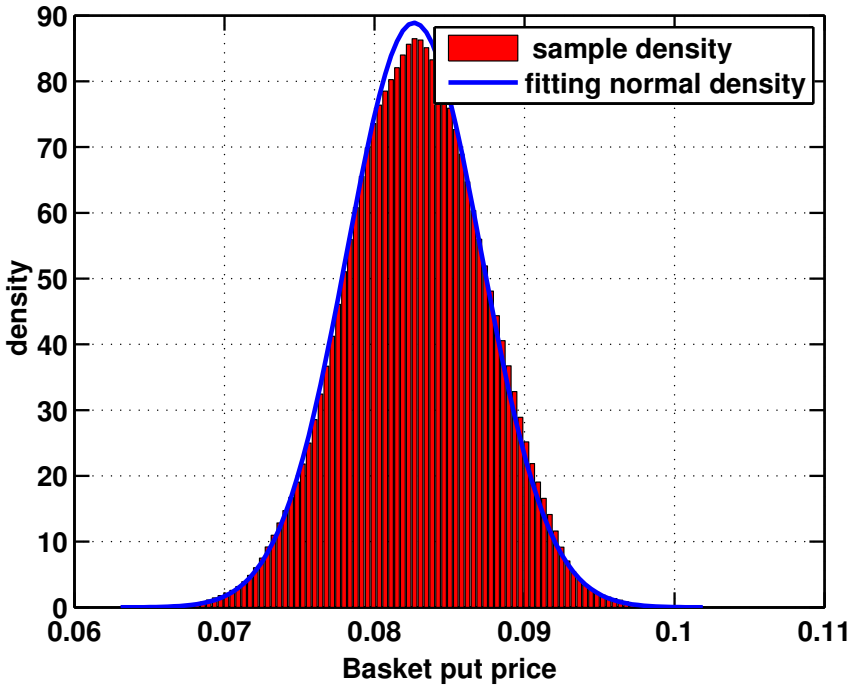



Figure 4.8: Density of six dimensional basket put price

Figure 4.8 shows the distribution of the basket put price in the case of six dimensional basket put. This was computed using (MC+Ju approximation ) and the kernel density estimation. We see that a Gaussian distribution with parameters $(0.0826, 0.0044)$ fits very well the sample density of the basket put price.

## 4.6 Conclusion

In this chapter, we investigated the impact of uncertainty in the covariance matrix of assets' returns on the basket put price under Black-Scholes framework. We modeled the randomness in the covariance matrix of assets's returns with Wishart distribution and we developed three ways that use this distribution to compute mean quantities and sample densities of basket put price: a nested MC simulation and two methods, based on MC and SGQ techniques, that use approximation for basket option price. We looked at two different problems by varying dimensionality.

In the case of four dimensional basket put, the different methods converge to the same value of basket put price mean with a difference of about $10^{-4}$ for the number of samples and SGQ levels needed in this case. Nested MC simulation is the worst method as it is very time consuming. The method based on SGQ perform significantly better as it needs less nodes($\sim$ 23000 pts) than MC/Approx ($10^5$ pts). The approximation of basket put price using the kernel density method indicates that it can be reasonably fitted by a Gaussian

distribution.

For a six dimensional basket put, the different methods converge to the same value of basket put price mean with a difference of about $10^{-4}$ for the number of samples and SGQ levels needed in this case. Again, nested MC simulation is the worst method as it is very time consuming. However, unlike the previous case the method based on SGQ is no longer the best as it needs more nodes ( $\sim$ 126000 pts) than MC/Approx ($10^5$ pts).
The approximation of basket put price using the kernel density method indicates that it can be reasonably fitted by a Gaussian distribution.

# Conclusion

## Summary

The aim of this project was to develop a quantitative framework to investigate the impact of parameters' uncertainty on option pricing. We focused on both European single asset and basket options under Black-Scholes framework. We now go through our findings and list the main results that we have obtained.

In the case of European single asset option, we investigated the impact of volatility's uncertainty on a put's price. After getting the volatility sample from historical data and fitting its density by the best parametric distribution, we implemented two probabilistic methods: MC and PC approximation to compute mean quantities, $\alpha$ bounds and sample densities of the put prices. Both methods gave same results but the surrogate PC/MC method was significantly faster. For the polynomial orders and the number of samples needed in our case, PC method was up to 3300 times faster. The developed probabilistic approach has the advantage of being applied with slight modification to any kind of options or pricing models.

In the case of basket options, we computed the effects of uncertainty in the covariance matrix of assets' returns on the basket put price. After modeling the randomness in the covariance matrix with Wishart distribution, we developed three ways to compute mean quantities and sample densities of basket put price: a nested MC simulation and two methods, based on MC and SGQ techniques, that use an approximation of basket option price. We looked at two different problems with four and six dimensional basket puts.

For both cases, the different methods approximate the same value of basket put price with a difference of order $10^{-4}$.
In the case of four dimensional basket put, the method based on SGQ perform significantly better as it needs less nodes ( $\sim$ 23000 pts) than MC/Approx ($10^5$ pts) to approximate the same value with a difference of about $10^{-4}$. However, this is no longer the case for six dimensional basket put where SGQ needs more nodes ( $\sim$ 125000 pts) than MC/Approx

($10^5$ pts).
The approximation of basket put price density using the kernel density method indicates that it can be reasonably fitted by a Gaussian distribution for both cases.

## Possible Extensions

We now suggest some directions for future work. Firstly, as mentioned before, the probabilistic approach that we developed in one dimensional case using MC and PC approximation can be applied with slight modification to any kind of options or pricing models. So, we can extend our work for other parametric pricing models such as stochastic volatility model and other types of options like American options.

Another important extension is to develop PC approximation for multidimensional options in order to investigate the performance of this probabilistic technique for higher dimensional problems.

Finally, constructing better dimension adaptive sparse grids may improve the performance of SGQ in the case of multidimensional option pricing problems.

# APPENDICES

# Appendix A: Comparison of Approximation Methods

| Correlation ($\rho$ ) | Beisser | Ju | Levy | Monte Carlo CV | StdDev |
|---|---|---|---|---|---|
| 0,1 | 20,12 | 21,77 | 22,06 | 21,62 | (0,0319) |
| 0,3 | 24,21 | 25,05 | 25,17 | 24,97 | (0,0249) |
| 0,5 | 27,63 | 28,01 | 28,05 | 27,97 | (0,0187) |
| 0,7 | 30,62 | 30,74 | 30,75 | 30,72 | (0,0123) |
| 0,8 | 31,99 | 32,04 | 32,04 | 32,03 | (0,0087) |
| 0,95 | 33,92 | 33,92 | 33,92 | 33,92 | (0,0024) |
| Dev | 0,7 | 0,071 | 0,203 | | |

Table 8: Varying the correlation

| Strike (K) | Beisser | Ju | Levy | Monte Carlo CV | StdDev |
|---|---|---|---|---|---|
| 50 | 54,16 | 54,31 | 54,34 | 54,28 | (0,0383) |
| 60 | 47,27 | 47,48 | 47,52 | 47,45 | (0,0375) |
| 70 | 41,26 | 41,52 | 41,57 | 41,50 | (0,0369) |
| 80 | 36,04 | 36,36 | 36,40 | 36,52 | (0,0363) |
| 90 | 31,53 | 31,88 | 31,92 | 31,85 | (0,0356) |
| 100 | 27,63 | 28,01 | 28,05 | 27,98 | (0,0350) |
| 110 | 24,27 | 24,67 | 24,70 | 24,63 | (0,0344) |
| 120 | 21,36 | 21,77 | 21,80 | 21,74 | (0,0338) |
| 130 | 18,84 | 19,26 | 19,28 | 19,22 | (0,0332) |
| 140 | 16,65 | 17,07 | 17,10 | 17,05 | (0,0326) |
| 150 | 14,75 | 15,17 | 15,19 | 15,15 | (0,0320) |
| Dev | 0,323 | 0,0310 | 0,065 | | |

Table 9: Varying the strike

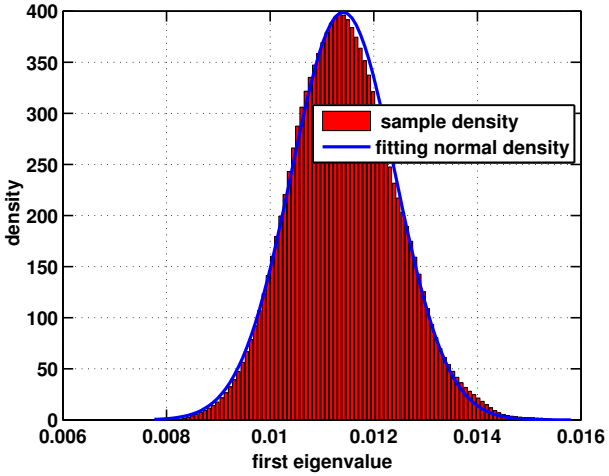
| S(0) | Beisser | Ju | Levy | Monte Carlo CV | StdDev |
|---|---|---|---|---|---|
| 50 | 4,16 | 4,34 | 4,34 | 4,34 | (0,0141) |
| 60 | 7,27 | 7,51 | 7,52 | 7,50 | (0,0185) |
| 70 | 11,26 | 11,55 | 11,57 | 11,53 | (0,0227) |
| 80 | 16,04 | 16,37 | 16,40 | 16,35 | (0,0268) |
| 90 | 21,53 | 21,89 | 21,92 | 21,86 | (0,0309) |
| 100 | 27,63 | 28,01 | 28,05 | 27,98 | (0,0350) |
| 110 | 34,27 | 34,66 | 34,70 | 34,63 | (0,0391) |
| 120 | 41,36 | 41,75 | 41,80 | 41,71 | (0,0433) |
| 130 | 48,84 | 49,23 | 49,28 | 49,19 | (0,0474) |
| 140 | 56,65 | 57,04 | 57,10 | 57,00 | (0,0516) |
| 150 | 64,75 | 65,13 | 65,19 | 65,08 | (0,0556) |
| Dev | 0,316 | 0,031 | 0,072 | | |

Table 10: Varying the Stock Prices

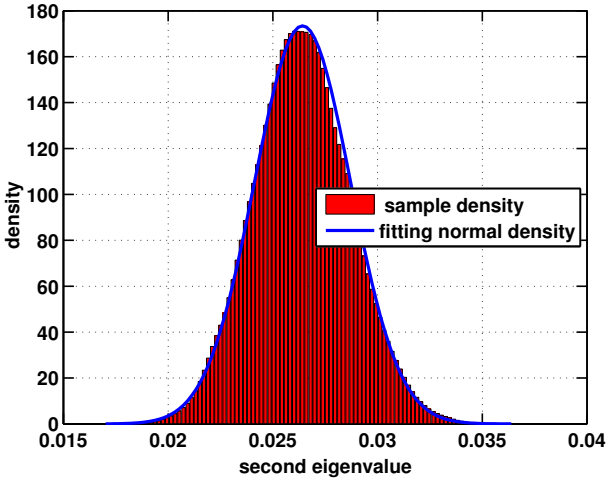
| Volatility | Beisser | Ju | Levy | Monte Carlo CV | StdDev |
|---|---|---|---|---|---|
| 5% | 3,53 | 3,53 | 3,53 | 3,53 | (0,0014) |
| 10% | 7,04 | 7,05 | 7,05 | 7,05 | (0,0042) |
| 15% | 10,55 | 10,57 | 10,57 | 10,57 | (0,0073) |
| 20% | 14,03 | 14,08 | 14,08 | 14,08 | (0,0115) |
| 30% | 20,91 | 21,08 | 21,09 | 21,07 | (0,0237) |
| 40% | 27,63 | 28,01 | 28,05 | 27,98 | (0,0350) |
| 50% | 34,15 | 34,84 | 34,96 | 34,80 | (0,0448) |
| 60% | 40,41 | 41,52 | 41,78 | 41,44 | (0,0327) |
| 70% | 46,39 | 47,97 | 48,50 | 47,86 | (0,0490) |
| 80% | 52,05 | 54,09 | 55,05 | 54,01 | (0,0685) |
| 100% | 62,32 | 64,93 | 67,24 | 65,31 | (0,0996) |
| Dev | 1,22 | 0,12 | 0,69 | | |

Table 11: Varying the volatility symetrically

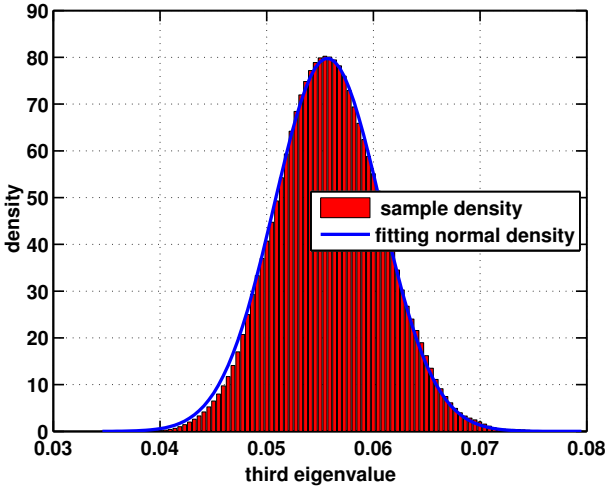
# Appendix B: Fitting Eigenvalues In Four Dimensional Case



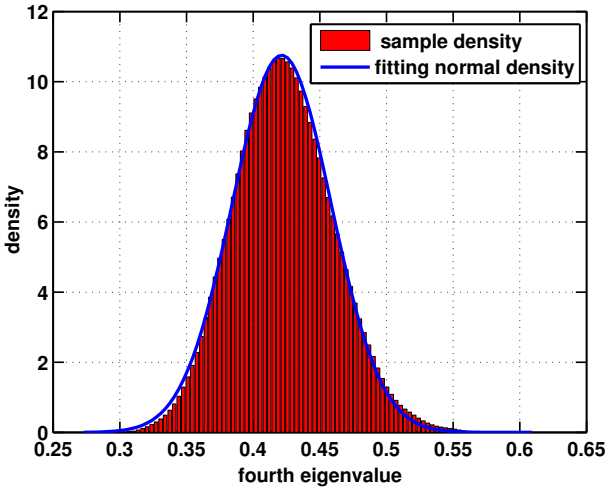

(a) Fitting first eigenvalue




(b) Fitting second eigenvalue

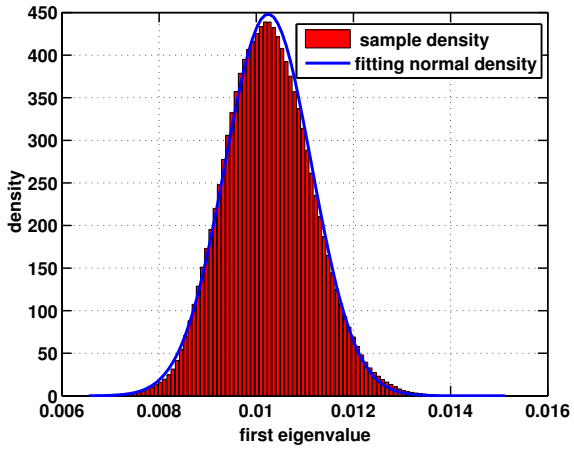



(a) Fitting third eigenvalue

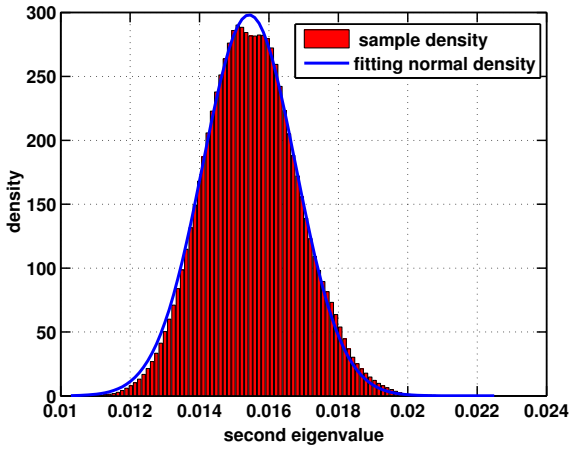



(b) Fitting fourth eigenvalue

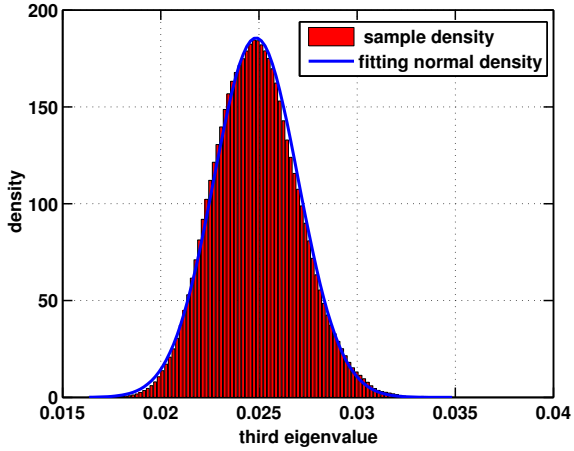
# Appendix C: Fitting Eigenvalues In six Dimensional Case



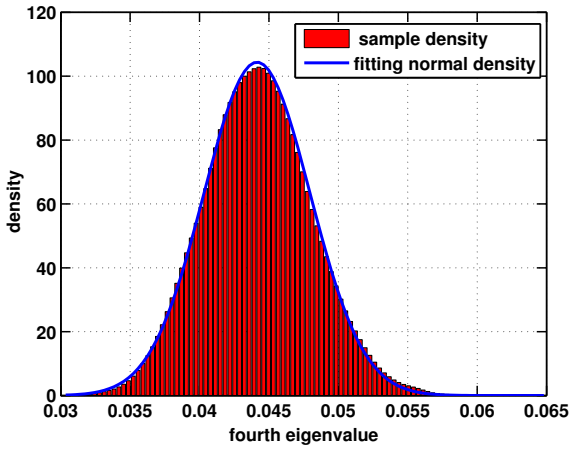

(a) Fitting first eigenvalue



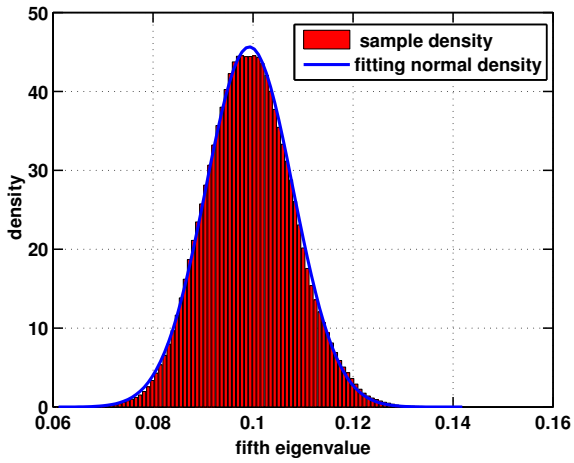

(b) Fitting second eigenvalue



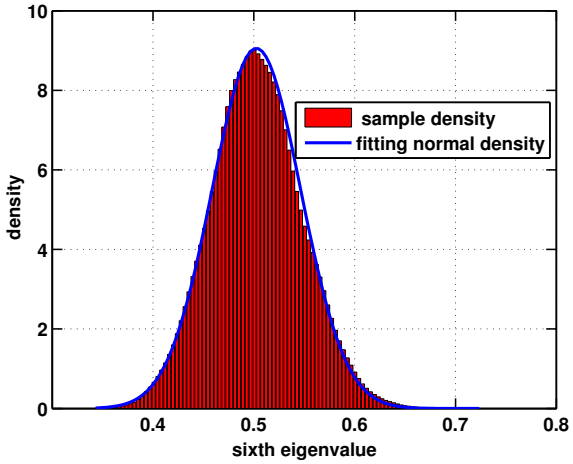

(a) Fitting third eigenvalue




(b) Fitting fourth eigenvalue




(a) Fitting fifth eigenvalue




(b) Fitting sixth eigenvalue

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