MEAN FIELD GAMES AND MEAN FIELD TYPE CONTROL THEORY

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August 16, 2013

OUTLINE I

OUTLINE II

ACKNOWLEDGEMENTS

Research supported by the Research Grants Council of Hong Kong Special Administrative Region (City U 500113)

GENERAL COMMENTS

- Mean field games reduces to a standard control problem and an equilibrium (fixed point)
  - Dynamic Programming: coupled HJB and FP equations
- Mean field type control is a non standard control problem.
  - Stochastic Maximum Principle (time inconsistency)
- Time inconsistency
- Major Playor
- Coalitions

MODEL

- Probability space $\Omega, \mathscr{A}, P$, filtration $\mathscr{F}^t$ generated by an n-dimensional standard Wiener process $w(t)$.
- The state space is R^n and the control space is R^d .

$$g(x,m,v) : R^n \times L^1(R^n) \times R^d \to R^n; \quad \sigma(x) : R^n \to \mathscr{L}(R^n;R^n)$$
$$f(x,m,v) : R^n \times L^1(R^n) \times R^d \to R; \quad h(x,m) : R^n \times L^1(R^n) \to R \tag{1}$$

$$\sigma(x), \sigma^{-1}(x) \text{ bounded} \tag{2}$$

- m is a probability density on R^n

STATE EQUATION

- $m(t) \in C(0,T;L^1(R^n))$ given . Feedback control $v(x,t)$.
- state of the system

$$dx = g(x(t), m(t), v(x(t))dt + \sigma(x(t))dw(t) \qquad (3)$$
$$x(0) = x_0$$

x_0 is a random variable independent of the Wiener process, probability density $m_0 = m(0)$.

- To the pair $v(.), m(.)$ we associate the control objective

$$J(v(.), m(.)) = E[\int_0^T f(x(t), m(t), v(x(t))\,dt + h(x(T), m(T)] \qquad (4)$$

MEAN FIELD GAME

- Find a pair $\hat{v}(.)$,$m(.)$ such that, denoting by $\hat{x}(.)$ the solution of

$$d\hat{x} = g(\hat{x}(t), m(t), \hat{v}(\hat{x}(t)))dt + \sigma(\hat{x}(t))dw(t) \tag{5}$$
$$\hat{x}(0) = x_0$$

then

$$m(t) \text{ is the probability distribution of } \hat{x}(t), \forall t \in [0, T] \tag{6}$$
$$J(\hat{v}(.), m(.)) \leq J(v(.), m(.)) \forall v(.)$$

MEAN FIELD TYPE CONTROL PROBLEM

- For any feedback $v(.)$, let $x(t) = x_{v(.)}(t)$ be the solution of (3) with $m(t)$ =probability distribution of $x_{v(.)}(t)$. So (3) becomes a McKean-Vlasov equation. Denote by $m_{v(.)}(t)$ =probability distribution of $x_{v(.)}(t)$, we thus have

$$dx_{v(.)} = g(x_{v(.)}(t), m_{v(.)}(t), v(x_{v(.)}(t))dt + \sigma(x_{v(.)}(t))dw(t) \quad (7)$$
$$x(0) = x_0$$

$$m_{v(.)}(t) = \text{probability distribution of}\, x_{v(.)}(t) \quad (8)$$

- Find $\hat{v}(.)$ such that

$$J(\hat{v}(.), m_{\hat{v}(.)}(.)) \leq J(v(.), m_{v(.)}(.)) \; \forall v(.) \quad (9)$$

NOTATION

Set

$$a(x) = \frac{1}{2}\sigma(x)\sigma^*(x) \tag{10}$$

and the 2nd order differential operator

$$A\varphi(x) = -\text{tr}\ a(x)D^2\varphi(x) \tag{11}$$

The Dual Operator is

$$A^*\varphi(x) = -\sum_{k,l=1}^{n} \frac{\partial^2}{\partial_{x_k}\partial_{x_l}}(a_{kl}(x)\varphi(x)) \tag{12}$$

GATEAUX DIFFERENTIABILITY

- Assume that the

$$m \to f(x,m,v),\ g(x,m,v),\ h(x,m) \tag{13}$$

are differentiable in $m \in L^2(R^n)$

- Notation $\dfrac{\partial f}{\partial m}(x,m,v)(\xi)$ to represent the derivative, so that

$$\frac{d}{d\theta} f(x, m+\theta\tilde{m}, v)_{|\theta=0} = \int_{R^n} \frac{\partial f}{\partial m}(x,m,v)(\xi)\tilde{m}(\xi)\,d\xi$$

OBJECTIVE FUNCTIONAL I

- Consider a feedback $v(x)$ and the corresponding trajectory defined by (7), the probability distribution $m_{v(.)}(t)$ of $x_{v(.)}(t)$ is solution of the FP equation

$$\frac{\partial m_{v(.)}}{\partial t} + A^* m_{v(.)} + \text{div}\,(g(x, m_{v(.)}, v(x))m_{v(.)}) = 0 \qquad (14)$$

$$m_{v(.)}(x,0) = m_0(x)$$

and the objective functional $J(v(.), m_{v(.)})$ can be expressed as follows

OBJECTIVE FUNCTIONAL II

$$J(v(.), m_{v(.)}(.)) = \int_0^T \int_{R^n} f(x, m_{v(.)}(x), v(x)) m_{v(.)}(x) dxdt + \tag{15}$$
$$+ \int_{R^n} h(x, m_{v(.)}(x, T)) m_{v(.)}(x, T) dx$$

FURTHER DIFFERENTIABILITY I

- Consider an optimal feedback $\hat{v}(x)$ and the corresponding probability density $m_{\hat{v}(.)}(x) = m(x)$.Let then $v(.)$ be any feedback and $\hat{v}(x) + \theta v(x)$. We want to compute

$$\frac{dm_{\hat{v}(.)+\theta v(.)}(x)}{d\theta}|_{\theta=0} = \tilde{m}(x)$$

We can check that

FURTHER DIFFERENTIABILITY II

$$\frac{\partial \tilde{m}}{\partial t} + A^* \tilde{m} + \text{div}\ (g(x,m,\hat{v}(x))\tilde{m}) + \tag{16}$$

$$+\text{div}([\int \frac{\partial g}{\partial m}(x,m,\hat{v}(x))(\xi)\tilde{m}(\xi)d\xi + \frac{\partial g}{\partial v}(x,m,\hat{v}(x))v(x)]m(x)) = 0$$

$$\tilde{m}(x,0) = 0$$

COST DIFFERENTIABILITY I

$$\frac{dJ(\hat{v}(.)+\theta v(.), m_{\hat{v}(x)+\theta v(x)}(.))}{d\theta}|_{\theta=0} = \tag{17}$$

$$\int_0^T \int_{R^n} f(x,m,\hat{v}(x))\tilde{m}(x)dtdx +$$

$$\int_0^T \int_{R^n}\int_{R^n} \frac{\partial f}{\partial m}(x,m,\hat{v}(x))(\xi)\tilde{m}(\xi)m(x)dtd\xi dx +$$

$$\int_0^T \int_{R^n} \frac{\partial f}{\partial v}(x,m,\hat{v}(x))v(x)m(x)dtdx +$$

COST DIFFERENTIABILITY II

$$+\int_{R^n} h(x, m(T))\tilde{m}(x, T)dx$$

$$+\int_{R^n}\int_{R^n}\frac{\partial h}{\partial m}(x, m(T))(\xi)\tilde{m}(\xi, T)m(x, T)d\xi dx$$

FUNCTION $u(x,t)$

- Introduce the function $u(x,t)$ solution of

$$-\frac{\partial u}{\partial t} + Au - g(x,m,\hat{v}(x)).Du - \int_{R^n} Du(\xi).\frac{\partial g}{\partial m}(\xi,m,\hat{v}(\xi))(x)m(\xi)d\xi$$
$$= f(x,m,\hat{v}(x)) + \int_{R^n} \frac{\partial f}{\partial m}(\xi,m,\hat{v}(\xi))(x)m(\xi)d\xi \tag{18}$$
$$u(x,T) = h(x,m(T)) + \int_{R^n} \frac{\partial h}{\partial m}(\xi,m(T))(x)m(\xi,T)d\xi$$

NECESSARY CONDITION I

$$\frac{dJ(\hat{v}(.)+\theta v(.), m_{\hat{v}(x)+\theta v(x)}(.))}{d\theta}|_{\theta=0} =$$

$$\int_0^T \int_{R^n} \frac{\partial f}{\partial v}(x,m,\hat{v}(x))v(x)m(x)dtdx +$$

$$+ \int_0^T \int_{R^n} Du(x).\frac{\partial g}{\partial v}(x,m,\hat{v}(x))v(x)m(x)dtdx$$

Since $\hat{v}(.)$ is optimal, this expression must vanish for any $v(.)$. Hence necessarily

$$\frac{\partial f}{\partial v}(x,m,\hat{v}(x)) + \frac{\partial g}{\partial v}^*(x,m,\hat{v}(x))Du(x) = 0 \tag{19}$$

REWRITING I

It follows that (at least with convexity assumptions)

$$\hat{v}(x) = \hat{v}(x, m, Du(x)) \tag{20}$$

We note that

$$f(x, m, \hat{v}(x)) + g(x, m, \hat{v}(x)).Du = H(x, m, Du) \tag{21}$$

$$\int_{R^n} [\frac{\partial f}{\partial m}(\xi, m, \hat{v}(\xi))(x) + Du(\xi).\frac{\partial g}{\partial m}(\xi, m, \hat{v}(\xi))(x)]m(\xi)d\xi = \tag{22}$$

$$\int_{R^n} \frac{\partial H}{\partial m}(\xi, m, Du(\xi))(x)m(\xi)d\xi$$

REWRITING II

$$g(x, m, \hat{v}(x)) = g(x, m, \hat{v}(x, m, Du(x))) = G(x, m, Du) \qquad (23)$$

HJB-FP SYSTEM

We can finally write the system of HJB-FP P.D.E.

$$
\begin{aligned}
-\frac{\partial u}{\partial t}+Au &= H(x,m,Du)+\int_{R^n}\frac{\partial H}{\partial m}(\xi,m,Du(\xi))(x)m(\xi)d\xi \\
u(x,T) &= h(x,m(T))+\int_{R^n}\frac{\partial h}{\partial m}(\xi,m(T))(x)m(\xi,T)d\xi \qquad (24)\\
\frac{\partial m}{\partial t}+A^*m+\text{div }(G(x,m,Du)m) &= 0 \\
m(x,0) &= m_0(x)
\end{aligned}
$$

SPIKE MODIFICATION I

- Instead of changing $\hat{v}(x,t)$ into $\hat{v}(x,t)+\theta v(x,t)$ one can use a spike modification

$$\bar{v}(x,s) = \left| \begin{array}{ll} v & s \in (t, t+\varepsilon) \\ \hat{v}(x,s) & s \notin (t, t+\varepsilon) \end{array} \right.$$

similar to the proof of Pontryagin maximum principle.

- One proves directly that $\hat{v}(x,t)$ minimizes the Lagrangian in v, instead of simply being a stationary point

NOTATION

From the optimal feedback $\hat{v}(x)$ and the probability distribution $m(t)$ we construct stochastic processes $X(t) \in R^n$, $V(t) \in R^d$, $Y(t) \in R^n$, $Z(t) \in \mathscr{L}(R^n; R^n)$ which are adapted, defined as follows

$$X(t) = \hat{x}(t),\ m(t) = P_{X(t)}$$

We next define

$$Y(t) = Du(X(t),t),\ V(t) = \hat{v}(X(t), P_{X(t)}, Y(t))$$

and finally

$$Z(t) = D^2 u\, \sigma(X(t),t)$$

STOCHASTIC MAXIMUM PRINCIPLE I

$$dX = g(X(t), P_{X(t)}, V(t))dt + \sigma(X(t))dw(t)$$

$$-dY = \Big(\frac{\partial H}{\partial x}(X(t), P_{X(t)}, V(t), Y(t)) + \qquad (25)$$

$$E[\frac{\partial^2 H}{\partial x \partial m}(X(t), P_{X(t)}, V(t), Y(t))](X(t)) + \mathrm{tr}\, \frac{\partial \sigma(X(t))}{\partial x}^* Z(t) \Big) dt$$

$$- Z(t)dw(t) \qquad (26)$$

$$X(0) = x_0, Y(T) = \frac{\partial h(X(T), P_{X(T)})}{\partial x} + E[\frac{\partial^2 h}{\partial x \partial m}(X(T), P_{X(T)}] \, (X(T)$$

STOCHASTIC MAXIMUM PRINCIPLE I

$$V(t) \text{ minimizes } H(X(t), P_{X(t)}, v, Y(t)) \text{ in } v \tag{27}$$

When we write

$$E[\frac{\partial^2 f}{\partial x \partial m}(X(t), P_{X(t)}, V(t))](X(t))$$

we mean that we take the function $\dfrac{\partial f}{\partial m}(\xi, m, v)(x)$, where ξ and v are parameters and we take the gradient in x, denoted by $\dfrac{\partial^2 f}{\partial x \partial m}(\xi, m, v)(x)$.
We then consider $\xi = X(t)$, $v = V(t)$ and take the expected value $E\dfrac{\partial^2 f}{\partial x \partial m}(X(t), m, V(t))(x)$.

STOCHASTIC MAXIMUM PRINCIPLE II

We take $m = P_{X(t)}$ (note that it is a deterministic quantity) and thus get $E\dfrac{\partial^2 f}{\partial x \partial m}(X(t), P_{X(t)}, V(t))(x)$.
Finally, we take the argument $x = X(t)$.

POSSIBLE CONFUSION I

To emphasize the difficulty of confusion, consider $\dfrac{\partial f}{\partial x}(x,m,v)$. If we want to take the derivative with respect to m, then we should consider x, v as parameters, so change the notation to ξ and compute $\dfrac{\partial^2 f}{\partial m \partial x}(\xi,m,v)(x)$.

Clearly

$$\frac{\partial^2 f}{\partial m \partial x}(\xi,m,v)(x) \neq \frac{\partial^2 f}{\partial x \partial m}(\xi,m,v)(x)$$

PARTICULAR CASE I

We discuss here the following particular mean field type problem

$$dx = g(x(t), v(x(t))dt + \sigma(x(t))dw(t) \tag{28}$$
$$x(0) = x_0$$

$$J(v(.), m(.)) = E[\int_0^T f(x(t), v(x(t))\, dt \ + h(x(T))] \tag{29}$$
$$+ \int_0^T F(Ex(t))dt + \Phi(Ex(T))$$

We consider a feedback $v(x,t)$ and $m(t) = m_{v(.)}(t)$ is the probability density of $x_{v(.)}(t)$ the solution of (28).

PARTICULAR CASE II

The functional becomes $J(v(.), m_{v(.)}(.))$. It is clearly a particular case of mean field type control problem.

NOTATION I

We have indeed

$$f(x,m,v)=f(x,v)+F(\int \xi m(\xi)d\xi)$$
$$h(x,m)=h(x)+\Phi(\int \xi m(\xi)d\xi)$$

Therefore

$$H(x,m,q)=H(x,q)+F(\int \xi m(\xi)d\xi)$$

where

$$H(x,q)=\inf_v(f(x,v)+q.g(x,v))$$

NOTATION II

Considering $\hat{v}(x,q)$ which attains the infimum in the definition of $H(x,q)$ and setting

$$G(x,q) = g(x,\hat{v}(x,q))$$

HJB-FP SYSTEM I

The coupled system HJB-FP becomes , see (24),

$$-\frac{\partial u}{\partial t}+Au=H(x,Du)+F(\int\xi m(\xi)d\xi)+\sum_k\frac{\partial F}{\partial x_k}(\int\xi m(\xi)d\xi)x_k$$
$$u(x,T)=h(x)+\Phi(\int\xi m(\xi)d\xi)+\sum_k\frac{\partial\Phi}{\partial x_k}(\int\xi m(\xi)d\xi)x_k \tag{30}$$

$$\frac{\partial m}{\partial t}+A^*m+\text{div }(G(x,Du)m)=0$$
$$m(x,0)=\delta(x-x_0)$$

REWRITING I

We can reduce slightly this problem, using the following step: introduce the vector function $\Psi(x,t;s)$, $t<s$, solution of

$$-\frac{\partial \Psi}{\partial t}+A\Psi-D\Psi.G(x,Du)=0, t<s \tag{31}$$
$$\Psi(x,s;s)=x$$

then

$$\int \xi m(\xi,t)d\xi=\Psi(x_0,0;t)$$

so (30) becomes

REWRITING II

$$-\frac{\partial u}{\partial t} + Au = H(x, Du) + F(\Psi(x_0, 0; t)) + \sum_k \frac{\partial F}{\partial x_k}(\Psi(x_0, 0; t))x_k$$

$$u(x, T) = h(x) + \Phi(\Psi(x_0, 0; T)) + \sum_k \frac{\partial \Phi}{\partial x_k}(\Psi(x_0, 0; T))x_k \quad (32)$$

PRECOMMITMENT

We now have the system (31), (32). We can also look at $u(x,t)$ as the solution of a non-local HJB equation, depending on the initial state x_0. The optimal feedback

$$\hat{v}(x,t) = \hat{v}(x, Du(x,t))$$

depends also on x_0. Note that it does not depend on any intermediate state. This type of optimal control is called a *pre-commitment* optimal control.

GAME CONCEPT

In [8], the authors introduce a new concept, in order to define an optimization problem among feedbacks which do not depend on the initial condition.

- A feedback will be optimal only against spike changes, but not against global changes.
- Game interpretation. Players are attached to small periods of time (eventually to each time, in the limit). Therefore, if one uses the concept of Nash equilibrium, decisions at different times correspond to decisions of different players, and thus out of reach.

NOTATION I

In the spirit of Dynamic Programming, and the invariant embedding idea, we consider a family of control problems indexed by the initial conditions, and we control the system using feedbacks only. So if $v(x,s)$ is a feedback, we consider the state equation $x(s) = x_{xt}(s; v(.))$

$$dx = g(x(s), v(x(s), s))ds + \sigma(x(s))dw(t) \tag{33}$$
$$x(t) = x$$

and the payoff

NOTATION II

$$J_{x,t}(v(.)) = E[\int_t^T f(x(s), v(x(s),s))\,ds + h(x(T)] + \tag{34}$$
$$+ \int_t^T F(Ex(s))ds + \Phi(Ex(T)))$$

Consider a specific control $\hat{v}(x,s)$ which will be optimal . We define $\hat{x}(.)$ to be the corresponding state, solution of (33) and set

$$V(x,t) = J_{x,t}(\hat{v}(.)) \tag{35}$$

SPIKE MODIFICATION I

We make a spike modification and define

$$\bar{v}(x,s) = \left| \begin{array}{cc} v & t < s < t+\varepsilon \\ \hat{v}(x,s) & s > t+\varepsilon \end{array} \right.$$

where v is arbitrary. The idea is to evaluate $J_{x,t}(\bar{v}(.))$ and to express that it is larger than $V(x,t)$. We introduce the function

$$\Psi(x,t;s) = E\hat{x}_{xt}(s),\ t < s$$

which is the solution of

SPIKE MODIFICATION II

$$-\frac{\partial \Psi}{\partial t}+A\Psi-D\Psi.g(x,\hat{v}(x,t))=0, t<s \tag{36}$$
$$\Psi(x,s;s)=x$$

We note the important property

$$E\bar{x}(s)=E\Psi(x(t+\varepsilon),t+\varepsilon;s), \forall s\geq t+\varepsilon$$

COMPARISON I

Therefore

$$J_{x,t}(\bar{v}(.)) = E[\int_t^{t+\varepsilon} f(x(s), v)ds+$$

$$+\int_{t+\varepsilon}^{T} f(\hat{x}_{x(t+\varepsilon),t+\varepsilon}(s), \hat{v}(\hat{x}_{x(t+\varepsilon),t+\varepsilon}(s), s))ds + h(\hat{x}_{x(t+\varepsilon),t+\varepsilon}(T))]$$

$$+\int_t^{t+\varepsilon} F(Ex(s))ds + \int_{t+\varepsilon}^{T} F(E\Psi(x(t+\varepsilon), t+\varepsilon; s))ds+$$

$$+\Phi(E\Psi(x(t+\varepsilon), t+\varepsilon; T)$$

COMPARISON I

The next point is to compare $F(E\Psi(x(t+\varepsilon),t+\varepsilon;s))$ with $EF(\Psi(x(t+\varepsilon),t+\varepsilon;s))$. This is a simple application of Ito's formula

$$EF(\Psi(x(t+\varepsilon),t+\varepsilon;s))-F(E\Psi(x(t+\varepsilon),t+\varepsilon;s))= \tag{37}$$

$$\varepsilon\sum_{ij}a_{ij}(x)\sum_{kl}\frac{\partial^2F}{\partial x_k\partial x_l}(\Psi(x,t;s))\frac{\partial\Psi_k}{\partial x_i}\frac{\partial\Psi_l}{\partial x_j}(x,t;s))+0(\varepsilon)$$

We can similarly compute the difference $E\Phi(\Psi(x(t+\varepsilon),t+\varepsilon;T))-\Phi(E\Psi(x(t+\varepsilon),t+\varepsilon;T))$.

EVALUATION OF THE PAYOFF I

$$J_{x,t}(\bar{v}(.)) = EV(x(t+\varepsilon), t+\varepsilon) + \varepsilon[f(x,v) + F(x) -$$

$$-\sum_{ij} a_{ij}(x) \int_t^T \sum_{kl} \frac{\partial^2 F}{\partial x_k \partial x_l}(\Psi(x,t;s)) \frac{\partial \Psi_k}{\partial x_i} \frac{\partial \Psi_l}{\partial x_j}(x,t;s))ds -$$

$$-\sum_{ij} a_{ij}(x) \sum_{kl} \frac{\partial^2 \Phi}{\partial x_k \partial x_l}(\Psi(x,t;T)) \frac{\partial \Psi_k}{\partial x_i} \frac{\partial \Psi_l}{\partial x_j}(x,t;T))] + 0(\varepsilon)$$

HJB EQUATION I

$$-\frac{\partial V}{\partial t}+AV=H(x,DV)+F(x)-$$

$$-\sum_{ijkl}a_{ij}(x)[\int_t^T\frac{\partial^2 F}{\partial x_k\partial x_l}(\Psi(x,t;s))\frac{\partial \Psi_k}{\partial x_i}\frac{\partial \Psi_l}{\partial x_j}(x,t;s))ds+ \tag{38}$$

$$+\frac{\partial^2 \Phi}{\partial x_k\partial x_l}(\Psi(x,t;T))\frac{\partial \Psi_k}{\partial x_i}\frac{\partial \Psi_l}{\partial x_j}(x,t;T))]$$

$$V(x,T)=h(x)+\Phi(x)$$

FUNCTION Ψ I

Moreover the equation for Ψcan be written as

$$-\frac{\partial \Psi}{\partial t} + A\Psi - D\Psi.G(x, DV) = 0, t < s \tag{39}$$

$$\Psi(x, s; s) = x$$

The optimal feedback obtained from the system (38), (39) is time consistent.

STATEMENT OF THE PROBLEM I

The mean-variance problem is the extension in continuous time for a finite horizon of the Markowitz optimal portfolio theory. Without referring to the background of the problem, it can be stated as follows, mathematically. The state equation is

$$dx = rxdt + xv.(\alpha dt + \sigma dw) \tag{40}$$

$$x(0) = x_0$$

$x(t)$ is scalar, r is a positive constant, α is a vector in R^m and σ is a matrix in $\mathscr{L}(R^d; R^m)$. All can depend on time and they are deterministic quantities. $v(t)$ is the control in R^m.
We note that, conversely to our general framework, the control affects the volatility term. The objective function is

STATEMENT OF THE PROBLEM II

$$J(v(.)) = Ex(T) - \frac{\gamma}{2}\text{var}(x(T)) \tag{41}$$

which we want to maximize.

MEAN FIELD TYPE CONTROL PROBLEM I

Because of the variance term, the problem is not a standard stochastic control problem. It is a mean field type control problem, since one can write

$$J(v(.)) = E(x(T) - \frac{\gamma}{2}x(T)^2) + \frac{\gamma}{2}(Ex(T))^2 \tag{42}$$

We consider a feedback control $v(x,s)$ and the corresponding state $x_{v(.)}(t)$ solution of (40) when the control is replaced by the feedback.

We associate the probability density $m_{v(.)}(x,t)$ solution of

MEAN FIELD TYPE CONTROL PROBLEM II

$$\frac{\partial m_{v(.)}}{\partial t} + \frac{\partial}{\partial x}(x m_{v(.)}(r + \alpha . v(x))) - \frac{1}{2}\frac{\partial^2}{\partial x^2}(x^2 m_{v(.)}|\sigma^* v(x)|^2) = 0 \tag{43}$$

$$m_{v(.)}(x,0) = \delta(x - x_0)$$

The functional (42) can be written as

$$J(v(.)) = \int m_{v(.)}(x,T)(x - \frac{\gamma}{2}x^2)dx + \frac{\gamma}{2}(\int m_{v(.)}(x,T)x dx)^2 \tag{44}$$

NECESSARY CONDITIONS I

Let $\hat{v}(x,t)$ be an optimal feedback, and $m(t) = m_{\hat{v}(.)}(t)$. Using the mean field type control approach, we get a pair $u(x,t)$,$m(x,t)$ satisfying

$$-\frac{\partial u}{\partial t} - xr\frac{\partial u}{\partial x} + \frac{1}{2}\frac{(\frac{\partial u}{\partial x})^2}{\frac{\partial^2 u}{\partial x^2}}\alpha^*(\sigma\sigma^*)^{-1}\alpha = 0 \tag{45}$$

$$u(x,T) = x - \frac{\gamma}{2}x^2 + \gamma x \int m(\xi,T)\xi d\xi$$

NECESSARY CONDITIONS II

$$\frac{\partial m}{\partial t} + r\frac{\partial (xm)}{\partial x} - \frac{\partial}{\partial x}\left(m\frac{\frac{\partial u}{\partial x}}{\frac{\partial^2 u}{\partial x^2}} \right)\alpha^*(\sigma\sigma^*)^{-1}\alpha - \tag{46}$$

$$-\frac{1}{2}\frac{\partial^2}{\partial x^2}\left(m\frac{(\frac{\partial u}{\partial x})^2}{(\frac{\partial^2 u}{\partial x^2})^2} \right)\alpha^*(\sigma\sigma^*)^{-1}\alpha = 0 \tag{47}$$

$$m(x,0) = \delta(x - x_0)$$

OPTIMAL FEEDBACK I

The optimal feedback is defined by

$$\hat{v}(x,t) = -\frac{\frac{\partial u}{\partial x}}{x\frac{\partial^2 u}{\partial x^2}}(\sigma\sigma^*)^{-1}\alpha \tag{48}$$

We can solve explicitly the system (45), (46). We look for

$$u(x,t) = -\frac{1}{2}P(t)x^2 + s(t)x + \rho(t) \tag{49}$$

We also define

$$q(t) = \int m(\xi,t)\xi d\xi \tag{50}$$

SOLUTION I

We obtain

$$P(t) = \gamma \exp \int_t^T (2r - \alpha^*(\sigma\sigma^*)^{-1}\alpha)d\tau \tag{51}$$

$$s(t) = (1 + \gamma q(T)) \exp \int_t^T (r - \alpha^*(\sigma\sigma^*)^{-1}\alpha)d\tau$$

$$\rho(t) = \int_t^T \frac{1}{2}\frac{s^2}{P}\alpha^*(\sigma\sigma^*)^{-1}\alpha(\tau)d\tau$$

We have to fix $q(T)$. Equation (46) becomes

SOLUTION II

$$\frac{\partial m}{\partial t} + r\frac{\partial (xm)}{\partial x} - \frac{\partial}{\partial x}\left(m(x-\frac{s}{P})\right)\alpha^*(\sigma\sigma^*)^{-1}\alpha \qquad (52)$$

$$-\frac{1}{2}\frac{\partial^2}{\partial x^2}\left(m(x-\frac{s}{P})^2\right)\alpha^*(\sigma\sigma^*)^{-1}\alpha = 0$$

$$m(x,0) = \delta(x-x_0)$$

OPTIMAL FEEDBACK

If we test this equation with x we obtain easily

$$q(T) = x_0 \exp \int_0^T r d\tau + \frac{1}{\gamma}[\exp \int_0^T \alpha^*(\sigma\sigma^*)^{-1}\alpha)d\tau - 1] \qquad (53)$$

This completes the definition of the function $u(x,t)$. The optimal feedback is defined by , see (48)

$$\hat{v}(x,t) = -(\sigma\sigma^*)^{-1}\alpha + \frac{1}{x}\frac{1+\gamma q(T)}{\gamma}\exp - \int_t^T r d\tau \qquad (54)$$

We see that this optimal feedback depends on the initial condition x_0.

TIME CONSISTENCY APPROACH I

If we take the time consistency approach, we consider the family of problems

$$dx = rxds + xv(x,s).(\alpha dt + \sigma dw),\ s > t \qquad (55)$$
$$x(t) = x$$

and the pay-off

$$J_{x,t}(v(.)) = E(x(T) - \frac{\gamma}{2}x(T)^2) + \frac{\gamma}{2}(Ex(T))^2 \qquad (56)$$

Denote by $\hat{v}(x,s)$ an optimal feedback and set $V(x,t) = J_{x,t}(\hat{v}(.))$.

FUNCTION Ψ

We define

$$\Psi(x,t;T) = E\hat{x}_{xt}(T)$$

where $\hat{x}_{xt}(s)$ is the solution of (55) for the optimal feedback.

FUNCTION V I

The function $\Psi(x,t;T)$ is the solution of

$$\frac{\partial\Psi}{\partial t}+\frac{\partial\Psi}{\partial x}(rx+x\hat{v}(x,t)^*\alpha)+\frac{1}{2}x^2\frac{\partial^2\Psi}{\partial x^2}|\sigma^*\hat{v}(x,t)|^2=0$$
$$\Psi(x,T;T)=x$$

We can write

$$V(x,t)=E(\hat{x}_{xt}(T)-\frac{\gamma}{2}\hat{x}_{xt}(T)^2)+\frac{\gamma}{2}(\Psi(x,t;T))^2$$

SPIKE MODIFICATION I

We consider a spike modification

$$\bar{v}(x,s) = \left| \begin{array}{cc} v & t < s < t+\varepsilon \\ \hat{v}(x,s) & s > t+\varepsilon \end{array} \right.$$

then

$$J_{x,t}(\bar{v}(.)) = E((\hat{x}_{x(t+\varepsilon),t+\varepsilon}(T) - \frac{\gamma}{2}\hat{x}_{x(t+\varepsilon),t+\varepsilon}(T)^2)$$
$$+\frac{\gamma}{2}(E\Psi(x(t+\varepsilon),t+\varepsilon;T))^2$$

where $x(t+\varepsilon)$ corresponds to the solution of (55) at time $t+\varepsilon$ for the feedback equal to the constant v.

APPROXIMATION I

We note that

$$EV(x(t+\varepsilon),t+\varepsilon) = E((\hat{x}_{x(t+\varepsilon),t+\varepsilon}(T) - \frac{\gamma}{2}\hat{x}_{x(t+\varepsilon),t+\varepsilon}(T)^2) + \frac{\gamma}{2}E(\Psi(x(t+\varepsilon),t+\varepsilon;T))^2$$

so we have to compare $(E\Psi(x(t+\varepsilon),t+\varepsilon;T))^2$ with $E(\Psi(x(t+\varepsilon),t+\varepsilon;T))^2$. We see easily that

$$(E\Psi(x(t+\varepsilon),t+\varepsilon;T))^2 - E(\Psi(x(t+\varepsilon),t+\varepsilon;T))^2 = = -\varepsilon x^2 \frac{\partial^2\Psi}{\partial x^2}(x,t;T)|\sigma^* v|^2 + 0(\varepsilon)$$

HJB EQUATION I

We obtain the HJB equation

$$\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}rx+\max_{v}[x\frac{\partial V}{\partial x}v^*\alpha+\frac{x^2}{2}(\frac{\partial^2 V}{\partial x^2}-\gamma\frac{\partial^2\Psi}{\partial x^2}(x,t;T))v^*\sigma\sigma^*v]=0 \tag{57}$$

$$V(x,T)=x$$

A direct checking shows that

$$V(x,t)=x\exp r(T-t)+\frac{1}{2\gamma}\int_t^T\alpha^*(\sigma\sigma^*)\alpha ds \tag{58}$$

$$\Psi(x,t;T)=x\exp r(T-t)+\frac{1}{\gamma}\int_t^T\alpha^*(\sigma\sigma^*)\alpha ds$$

HJB EQUATION II

and

$$\hat{v}(x,t) = \frac{\exp -r(T-t)}{x\gamma}(\sigma\sigma^*)\alpha \tag{59}$$

This optimal control satisfies the time consistency property (it does not depend on the initial condition).

GENERAL CONSIDERATIONS I

In the preceding slides, we have considered a single population, composed of a large number of individuals, with identical behavior. In real situations, we will have several populations. The natural extension to the preceding developments is to obtain mean field equations for each population. A much more challenging situation will be to consider competing populations. We present first the approach of multi-class agents, as described in [16], [18].

MODEL I

Instead of functions $f(x,m,v)$,$g(x,m,v),h(x,m),\sigma(x)$ we consider K functions $f_k(x,m,v)$,$g_k(x,m,v),h_k(x,m),\sigma_k(x)$, $k=1,\cdots K$. The index k represents some characteristics of the agents, and a class corresponds to one value of the characteristics. So there are K classes. In the model discussed previously, we have considered a single class. In the sequel, when we consider an agent i,he will have a characteristics $\alpha^i \in (1,\cdots,K)$. Agents will be defined with upper indices, so $i=1,\cdots N$ with N very large. α^i is a known information. The important assumption is

$$\frac{1}{N}\sum_{i=1}^{N} 1\!\!1_{\alpha^i=k} \to \pi_k, \text{ as } N \to +\infty \tag{60}$$

MODEL II

and π_k is a probability distribution on the finite set of characteristics, which represents the probability that an agent has the characteristics k.

FURTHER NOTATION I

Generalizing the case of a single class, we define $a_k(x) = \frac{1}{2}\sigma_k(x)\sigma_k(x)^*$and the operator

$$A_k\varphi(x) = -\mathrm{tr} a_k(x) D^2\varphi(x)$$

We define Lagrangians, Hamiltonians indexed by k,namely

$$L_k(x,m,v,q) = f_k(x,m,v) + q.g_k(x,m,v)$$

$$H_k(x,m,q) = \inf_v L_k(x,m,v,q)$$

and $\hat{v}_k(x,m,q)$ denotes the minimizer in the defintion of the Hamiltonian. We also define

FURTHER NOTATION II

$$G_k(x,m,q) = g_k(x,m,\hat{v}_k(x,m,q))$$

SYSTEM OF HJB EQUATIONS I

Given a function $m(t)$ we consider the HJB equations, indexed by k

$$-\frac{\partial u_k}{\partial t} + Au_k = H_k(x, m, Du_k) \tag{61}$$
$$u_k(x, T) = h_k(x, m(T))$$

and the FP equations

$$\frac{\partial m_k}{\partial t} + A^* m_k + \mathrm{div}\,(G_k(x, m, Du_k)m_k) = 0 \tag{62}$$
$$m_k(x, 0) = m_{k0}(x) \tag{63}$$

SYSTEM OF HJB EQUATIONS II

in which the probability densities m_{k0} are given. A mean field game equilibrium for the multi class agents problem is attained whenever

$$m(x,t) = \sum_{k=1}^{K} \pi_k m_k(x,t), \ \forall x,t \tag{64}$$

GENERAL COMMENTS I

We consider here a problem initiated by Huang [15], in the L.Q. case. In a recent paper Nourian and Caines [23] have studied a non linear mean field game with a major player. In both papers, there is a simplification in the coupling between the major player and the representative agent. We will describe here the problem in full generality and explain the simplification which is done in [23]. The new element is that, besides the representative agent there is a major player. This major player influences directly the mean field term. Since the mean field term also impacts the major playor, he will takes this into account to define his decisions. On the other hand, the mean field term can no longer be deterministic, since it depends on the major player decisions. This coupling creates new difficulties.

MODEL OF MAJOR PLAYER I

We introduce the following state evolution for the major player

$$dx_0 = g_0(x_0(t), m(t), v_0(t))dt + \sigma_0(x_0)dw_0 \qquad (65)$$
$$x_0(0) = \xi_0$$

We assume that $x_0(t) \in \mathbb{R}^{n_0}$, $v_0(t) \in \mathbb{R}^{d_0}$. The process $w_0(t)$ is a standard Wiener process with values in $\mathbb{R}^{k_0}$ and ξ_0 is a random variable in $\mathbb{R}^{n_0}$ independent of the Wiener process. The process $m(t)$ is the mean field term, with values in the space of probabilities on $\mathbb{R}^n$. This term will come from the decisions of the representative agent.
However, It will be linked to $x_0(t)$ since the major player influences the decision of the representative agent.

MODEL OF MAJOR PLAYER II

If we define the filtration

$$\mathscr{F}^{0t} = \sigma(\xi_0, w_0(s), s \leq t) \tag{66}$$

then $m(t)$ is a process adapted to $\mathscr{F}^{0t}$. But it is not external.

MODEL OF MAJOR PLAYER I

We will describe the link with the state x_0 in analyzing the representative agent problem. The control $v_0(t)$ is also adapted to $\mathscr{F}^{0t}$. The objective functional of the major player is

$$J_0(v_0(.)) = E[\int_0^T f_0(x_0(t), m(t), v_0(t))dt + \\ +h_0(x_0(T), m(T))] \tag{67}$$

The functions g_0, f_0, σ_0, h_0 are deterministic. We do not specify the assumptions, since our treatment is formal.

MODEL OF REPRESENTATIVE AGENT I

The representative agent has state $x(t) \in \mathbb{R}^n$ and control $v(t) \in \mathbb{R}^d$.We have the evolution

$$dx = g(x(t), x_0(t), m(t), v(t))dt + \sigma(x(t))dw \tag{68}$$
$$x(0) = \xi$$

in which $w(t)$ is a standard Wiener process with values in $\mathbb{R}^k$ and ξ is a random variable with values in $\mathbb{R}^n$ independent of $w(.)$. Moreover, $\xi, w(.)$ are independent of $\xi_0, w_0(.)$. We define

$$\mathscr{F}^t = \sigma(\xi, w(s), s \leq t) \tag{69}$$
$$\mathscr{G}^t = \mathscr{F}^{0t} \cup \mathscr{F}^t \tag{70}$$

MODEL OF REPRESENTATIVE AGENT II

The control $v(t)$ is adapted to $\mathcal{G}^t$. The objective functional of the representative agent is defined by

$$J(v(.),x_0(.),m(.)) = E[\int_0^T f(x(t),x_0(t),m(t),v(t))dt + \\ +h(x(T),x_0(T),m(T))] \quad (71)$$

Conversely to the major player problem, in the representative agent problem, the processes $x_0(.)$,$m(.)$ are external. In (67) $m(t)$ depends on $x_0(.)$.

CONDITIONAL PROBABILITY DENSITY OF THE REPRESENTATIVE AGENT I

The representative agent's problem is similar to the standard situation except for the presence of $x_0(t)$.
We begin by limiting the class of controls for the representative agent to belong to feedbacks $v(x,t)$ random fields adapted to $\mathscr{F}^{0t}$. The corresponding state, solution of (68) is denoted by $x_{v(.)}(t)$. Of course, this process depends also of $x_0(t), m(t)$. Note that $x_0(t), m(t)$ is independent from $\mathscr{F}^{t}$, therefore the conditional probability density of $x_{v(.)}(t)$ given the filtration $\cup_t \mathscr{F}^{0t}$ is the solution of the F.P. equation with random coefficients

CONDITIONAL PROBABILITY DENSITY OF THE REPRESENTATIVE AGENT II

$$\frac{\partial p_{v(.)}}{\partial t} + A^* p_{v(.)} + \mathrm{div}(g(x, x_0(t), m(t), v(x,t)) p_{v(.)}) = 0 \tag{72}$$

$$p_{v(.)}(x, 0) = \varpi(x)$$

in which $\varpi(x)$ is the density probability of ξ.

OBJECTIVE FUNCTIONAL OF THE REPRESENTATIVE AGENT I

We can then rewrite the objective functional $J(v(.),x_0(.),m(.))$ as follows

$$J(v(.),x_0(.),m(.)) = E[\int_0^T \int_{\mathbb{R}^n} p_{v(.),x_0(.),m(.)}(x,t)f(x,x_0(t),m(t),v(x,t))d \tag{73}$$

$$+\int_{\mathbb{R}^n} p_{v(.),x_0(.),m(.)}(x,T)h(x,x_0(T),m(T))dx]$$

We can give an expression for this functional. Introduce the random field $\chi_{v(.)}(x,t)$ solution of the stochastic backward PDE:

OBJECTIVE FUNCTIONAL OF THE REPRESENTATIVE AGENT II

$$-\frac{\partial \chi_{v(.)}}{\partial t} + A\chi_{v(.)} = f(x, x_0(t), m(t), v(x,t)) + g(x, x_0(t), m(t), v(x,t)).D\chi \tag{74}$$

$$\chi_{v(.)}(x, T) = h(x, x_0(T), m(T))$$

then we can assert that

$$\int_0^T \int_{\mathbb{R}^n} p_{v(.),x_0(.),m(.)}(x,t) f(x, x_0(t), m(t), v(x,t)) dxdt + \int_{\mathbb{R}^n} p_{v(.),x_0(.),m(.)}(x,T) h(x, x_0(T), m(T)) dx = \int_{\mathbb{R}^n} \chi_{v(.)}(x,0)\varpi(x$$

COMPUTING THE OBJECTIVE FUNCTION I

We get

$$J(v(.),x_0(.),m(.)) = \int_{\mathbb{R}^n} \varpi(x) E\chi_{v(.)}(x,0)dx \tag{75}$$

Now define

$$u_{v(.)}(x,t) = E^{\mathscr{F}^{0t}} \chi_{v(.)}(x,t)$$

From equation (74) we can assert that

$$-E^{\mathscr{F}^{0t}} \frac{\partial \chi_{v(.)}}{\partial t} + Au_{v(.)} = f(x,x_0(t),m(t),v(x,t)) + g(x,x_0(t),m(t),v(x,t) \tag{76}$$

$$u_{v(.)}(x,T) = h(x,x_0(T),m(T))$$

BACKWARD SPDE I

On the other hand

$$u_{v(.)}(x,t) - \int_0^t E^{\mathscr{F}^{0s}} \frac{\partial \chi_{v(.)}}{\partial s}(x,s)ds$$

is a $\mathscr{F}^{0t}$ martingale. Therefore we can write

$$u_{v(.)}(x,t) - \int_0^t E^{\mathscr{F}^{0s}} \frac{\partial \chi_{v(.)}}{\partial s}(x,s)ds = u_{v(.)}(x,0) + \int_0^t K_{v(.)}(x,s)dw_0(s)$$

where $K_{v(.)}(x,s)$ is $\mathscr{F}^{0s}$ measurable, and uniquely defined. It is then easy to check that the random field $u_{v(.)}(x,t)$ is solution of the backward stochastic PDE (BSPDE) :

BACKWARD SPDE II

$$-\partial_t u_{v(.)}(x,t) + Au_{v(.)}(x,t)dt = f(x, x_0(t), m(t), v(x,t))dt + \tag{77}$$

$$+g(x, x_0(t), m(t), v(x,t)).Du_{v(.)}(x,t)dt - K_{v(.)}(x,t)dw_0(t)$$

$$u_{v(.)}(x,T) = h(x, x_0(T), m(T))$$

NECESSARY CONDITION I

From (75) we get immediately

$$J(v(.),x_0(.),m(.)) = \int_{\mathbb{R}^n} \varpi(x) E u_{v(.)}(x,0)dx \tag{78}$$

We then write a necessary condition of optimality for a control $\hat{v}(x,t)$.Setting $u(x,t) = u_{\hat{v}(.)}(x,t)$, $K(x,t) = K_{\hat{v}(.)}(x,t)$ we obtain the stochastic HJB equation

$$-\partial_t u(x,t) + Au(x,t)dt = H(x,x_0(t),m(t),Du)dt - K(x,t)dw_0 \tag{79}$$

$$u(x,T) = h(x,x_0(T),m(T))$$

and

NECESSARY CONDITION II

$$\hat{v}(x,t) = \hat{v}(x, x_0(t), m(t), Du(x,t)) \tag{80}$$

FP EQUATION I

We next have to express the mean field game condition

$$m(t) = p_{\hat{v}(.),x_0(.),m(.)}(.,t)$$

we obtain from (72) the FP equation

$$\frac{\partial m}{\partial t} + A^* m + \mathrm{div}(G(x, x_0(t), m(t), Du(x,t))m) = 0 \qquad (81)$$

$$m(x,0) = \varpi(x)$$

The coupled pair of HJB-FP equations (79),(81) allow to define the reaction function of the representative agent to the trajectory $x_0(.)$ of the major player. One defines the random fields $u(x,t), m(x,t)$ and the optimal feedback is given by (80).

MAJOR PLAYER I

Consider now the problem of the major player. In [23] and also [15] for the L.Q. case it is limited to (65), (67) since $m(t)$ is external. However since $m(t)$ is coupled to $x_0(t)$ through equations (79), (81) one cannot consider $m(t)$ as external, unless limiting the decision of the major player. So in fact the major player has to consider three state equations (65), (79), (81). For a a given $v_0(.)$ adapted to $\mathscr{F}^{0t}$ we associate $x_{0,v_0(.)}(.)$, $u_{v_0(.)}(.,.), m_{v_0(.)}(.,.)$ solution of the system (65), (79), (81).

Introduce the notation

MAJOR PLAYER II

$$
\begin{aligned}
H_0(x_0,m,p) &= \inf_{v_0}[f_0(x_0,m,v_0)+p.g_0(x_0,m,v_0)] \\
\hat{v}_0(x_0,m,p) & \quad \text{minimizes the expression in brackets} \\
G_0(x,m,p) &= g_0(x_0,m,\hat{v}_0(x_0,m,p))
\end{aligned}
$$

NECESSARY CONDITIONS FOR THE MAJOR PLAYER I

We have 3 adjoint equations

$$-dp = [H_{0,x_0}(x_0(t), m(t), p(t)) + \sum_{l=1}^{k_0} \sigma^*_{0l,x_0}(x_0(t))q_l(t) \\ + \int G^*_{x_0}(x, x_0(t), m(t), Du(x,t))D\eta(x,t)m(x,t)dx + \\ \int \zeta(x,t)H_{x_0}(x, x_0(t), m(t), Du(x,t)dx]dt - \sum_{l=1}^{k_0} q_l dw_{0l} \tag{82}$$

NECESSARY CONDITIONS FOR THE MAJOR PLAYER II

$$p(T) = h_{0,x_0}(x_0(T), m(T)) + \int \zeta(x,T) h_{x_0}(x, x_0(T), m(T)) dx$$

NECESSARY CONDITIONS FOR THE MAJOR PLAYER I

$$-\partial_t \eta + A\eta(x,t)dt = [\frac{\partial H_0}{\partial m}(x_0(t), m(t), p(t))(x)$$

$$+D\eta(x,t).G(x, x_0(t), m(t), Du(x,t))+$$
$$+\int D\eta(\xi,t).\frac{\partial G}{\partial m}(\xi, x_0(t), m(t), Du(\xi,t))(x)m(\xi,t)d\xi \qquad (83)$$

$$+\int \zeta(\xi,t)\frac{\partial H}{\partial m}(\xi, x_0(t), m(t), Du(\xi,t))(x)d\xi]dt \quad - \quad \sum_l \mu_l(x,t)dw_{0l}(t)$$

$$\eta(x,T) = \frac{\partial h_0}{\partial m}(x_0(T), m(T))(x) + \int \zeta(\xi,T)\frac{\partial h}{\partial m}(\xi, x_0(T), m(T))(x)d\xi$$

NECESSARY CONDITIONS FOR THE MAJOR PLAYER I

$$\frac{\partial \zeta}{\partial t} + A^*\zeta(x,t) + \text{div}\,(G(x,x_0(t),m(t),Du(x,t))\zeta(x,t)) \\ + \text{div}(G_q^*(x,x_0(t),m(t),Du(x,t))D\eta(x,t)\,m(x,t)) = 0 \quad (84)$$

$$\zeta(x,0) = 0$$

Next $x_0(t)$ satisfies

$$\begin{aligned} dx_0 &= G_0(x_0(t),m(t),p(t))dt + \sigma_0(x_0(t))dw_0 \quad (85) \\ x_0(0) &= \xi_0 \end{aligned}$$

NECESSARY CONDITIONS FOR THE MAJOR PLAYER II

So, in fact the complete solution is provided by the 6 equations (85),(82), (79), (84), (81), (83) and the feedback of the representative agent and the contol of the major player are given by (80) and

$$\hat{v}_0(t) = \hat{v}_0(x_0(t), m(t), p(t)) \tag{86}$$

SYSTEM OF HJB-FP EQUATIONS I

We can introduce more general problems

$$-\frac{\partial u^i}{\partial t} + Au^i = H^i(x, m, Du) \tag{87}$$
$$u^i(x, T) = h^i(x, m(T))$$

$$\frac{\partial m^i}{\partial t} + A^* m^i + \text{div}\,(G^i(x, m, Du)m^i) = 0 \tag{88}$$
$$m^i(x, 0) = m^i_0(x)$$

in which $m = (m^1, \cdots, m^N)$ and the functions H^i, G^i depend on the full vector m. The interpretation is much more elaborate.

DESCRIPTION OF THE GAME I

We want to associate to problem (87), (88) a differential game for N communities, composed of very large numbers of agents. We denote the agents by the index i,j where $i=1,\cdots N$ and $j=1,\cdots M$. The number M will tend to $+\infty$. Each player i,jchooses a feedback $v^{i,j}(x)$, $x\in\mathbb{R}^n$. The state of player i,j is denoted by $x^{i,j}(t)\in\mathbb{R}^n$. We consider independent standard Wiener processes $w^{i,j}(t)$ and independent replicas $x_0^{i,j}$ of the random variable x_0^i, whose probability density is m_0^i. They are independent of the Wiener processes. We denote

$$v^{.j}(.)=(v^{1,j}(.),\cdots,v^{N,j}(.))$$

The trajectory of the state $x^{i,j}$ is defined by the equation

DESCRIPTION OF THE GAME II

$$dx^{i,j} = g^i(x^{i,j}, v^{.j}(x^{i,j}))dt + \sigma(x^{i,j})dw^{i,j} \tag{89}$$
$$x^{i,j}(0) = x_0^{i,j}$$

DESCRIPTION OF THE GAME I

The trajectories are independent. The player i,j trajectory is influenced by the feedbacks $v^{k,j}(x)$, $k \neq i$ acting on his own state. When we focus on player i we use the notation

$$v^{.j}(.) = (v^{i,j}(.), \bar{v}^{i,j}(.))$$

in which $\bar{v}^{i,j}(.)$ represents all feedbacks $v^{k,j}(x)$, $k \neq i$. The notation $v(.)$ represents all feedbacks.

We now define the objective functional of player i,j. It is given by

DESCRIPTION OF THE GAME II

$$\mathscr{J}^{i,j}(v(.)) = E\int_0^T [f^i(x^{i,j}(t), v^{.j}(x^{i,j}(t))) + \tag{90}$$

$$f_0^i(x^{i,j}(t), \frac{1}{M-1}\sum_{l=1\neq j}^{M}\delta_{x^{i,l}(t)}))]dt + Eh^i(x^{i,j}(T), \frac{1}{M-1}\sum_{l=1\neq j}^{M}\delta_{x^{i,l}(T)})$$

We look for a Nash equilibrium.

APPROXIMATE NASH EQUILIBRIUM I

Consider next the system of pairs of HJB-FP equations (87), (88) and the feedback $\hat{v}(x)$.
We can show that the feedback

$$\hat{v}^{i,j}(.) = \hat{v}^{i}(.)$$

is an approximate Nash equilibrium.
If we use this feedback in the state equation (89) we get

$$\begin{aligned} d\hat{x}^{i,j} &= g^{i}(\hat{x}^{i,j}, \hat{v}(\hat{x}^{i,j}))dt + \sigma(\hat{x}^{i,j})dw^{i,j} \\ \hat{x}^{i,j}(0) &= x_0^{i,j} \end{aligned}$$

APPROXIMATE NASH EQUILIBRIUM II

and the trajectories $\hat{x}^{i,j}$ become independent replicas of $\hat{x}^i$solution of

$$\begin{aligned} d\hat{x}^i &= g^i(\hat{x}^i,\hat{v}(\hat{x}^i))dt+\sigma(\hat{x}^i)dw^i \\ \hat{x}^i(0) &= x_0^i \end{aligned}$$

APPROXIMATE NASH EQUILIBRIUM I

The probability density of $\hat{x}^i(t)$ is $m^i(t)$. We first prove

$$\mathscr{J}^{i,j}(\hat{v}(.)) - J^i(\hat{v}(.), m^i(.)) \to 0, \text{ as } M \to +\infty.$$

We now focus on player 1,1 to fix the ideas. Suppose he uses a feedback $v^{1,1}(x) \neq \hat{v}^{1,1}(x)$, and the other players use $\hat{v}^{i,j}(x) = \hat{v}^i(x), \forall i \geq 2, \forall j$ or $\forall i, \forall j \geq 2$. We set $v^1(x) = v^{1,1}(x)$. Call this set of controls $\tilde{v}(.)$. By abuse of notation, we also write

$$\tilde{v}(.) = (v^1(.), \hat{v}^2(.), \cdots \hat{v}^N(.)) = (v^1(.), \overline{\hat{v}}^1(.))$$

The corresponding trajectories are denoted by $y^{1,j}(t)$ solutions of

APPROXIMATE NASH EQUILIBRIUM II

$$dy^{1,1} = g^1(y^{1,1}, v^1(y^{1,1}), \overline{\hat{v}}^1(y^{1,1}))dt + \sigma(y^{1,1})dw^{1,1} \tag{91}$$
$$y^{1,1}(0) = x_0^{1,1}$$

and $y^{1,j} = \hat{x}^{1,j}$ for $j \geq 2$.
We can then prove that

$$\mathscr{J}^{1,1}(\tilde{v}(.)) \geq J^1(\hat{v}(.), m^1(.)) - O(M)$$

and this concludes the approximate Nash equilibrium property.

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Thanks!