Virtual Element Spaces and Applications

F. Brezzi

IUSS-Istituto Universitario di Studi Superiori, Pavia, Italy

IMATI-C.N.R., Pavia, Italy

KAUST, April 28-th, 2014

Outline

Find $u \in V \equiv H^1_0(\Omega)$ *s. t.* $-\Delta u = f$. That is:

$$\int_\Omega \nabla u \cdot \nabla v \, d\Omega = \int_\Omega f \, v \, d\Omega \quad \forall \, v \in V.$$

where Ω is, say, a polygon in $\mathbb{R}^2$ and $f \in L^2(\Omega)$ is given.

For $N \in \mathbb{N}$ we would like to construct an $N \times N$ nonsingular matrix Δ_N and a vector $F_N \in \mathbb{R}^N$ such that the solution $U_N \in \mathbb{R}^N$ of the linear system

$$-\Delta_N \, U_N = F_N$$

is *an approximation* (in a sense to be made precise!) of the exact solution u (better and better as N grows).

Two big classes of methods

Very roughly, the (zillions of) methods available on the market can be split in two categories:

- Every N-ple $V_N \in \mathbb{R}^N$ is uniquely associated to a function $v_N(x,y) \in H^1_0(\Omega)$, and u_N (corresponding to the discrete solution $U_N \in \mathbb{R}^N$) is an approximation of the exact solution u. (FEM, Spectral Methods, RBF, XFEM, etc. Now also VEM)
- We have a linear functional χ_N from $C^0(\overline{\Omega}) \cap H^1_0(\Omega)$ into $\mathbb{R}^N$ (e.g. point values), and we require that $\|U_N - \chi_N(u)\| \to 0$ in some suitable norm. (FD, FV, MFD, Cochains, etc.)

Given a *triangulation* $\mathcal{T}_h$ of Ω, with N internal nodes, we set $V_h =$ continuous piecewise linear functions vanishing on $\partial\Omega$, and we look for u_h in V_h such that

$$\int_\Omega \nabla u_h \cdot \nabla v_h \, d\Omega = \int_\Omega f \, v_h \, d\Omega \quad \forall \, v_h \in V_h.$$

Here $U_N =$ values of u_h at the nodes.

Note: the final matrix is usually computed as the sum of the contributions of the single elements:

$$A_{i,j} \equiv \int_\Omega \nabla v^j \cdot \nabla v^i \, d\Omega = \sum_{E \in \mathcal{T}_h} \int_E \nabla v^j \cdot \nabla v^i \, dE.$$

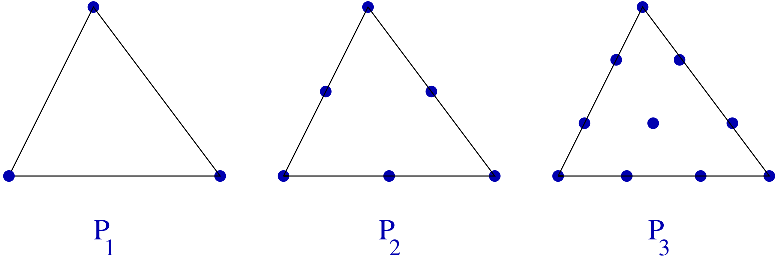
More generally, in FEM the degrees of freedom are used to reconstruct polynomials (or isoparametric images of polynomials) in each element.

Ingredients:

- the geometry of the element (e.g.: triangles)
- the degrees of freedom; say, n d.o.f. per element
- in each element, a space of polynomials of dim. n.

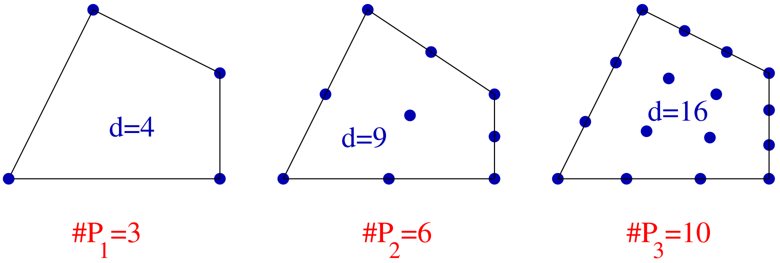
The ingredients must match

- *Unisolvence* n numbers $\leftrightarrow$ one and only one polynomial
- *Continuity*



Nodal values. C^0 continuity

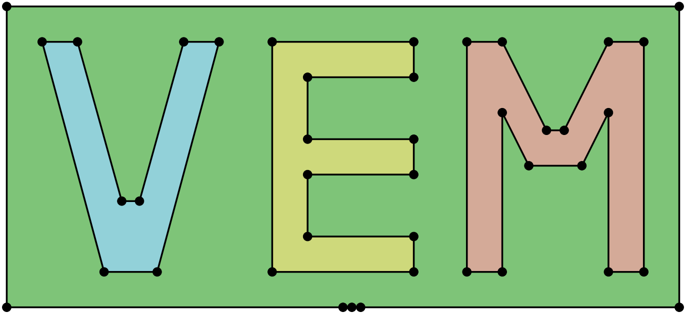
Traditional finite elements-Quads



Nodal values. C^0 continuity

A flavor of VEMs

For a decomposition in more general sub-polygons, FEM face considerable difficulties. With VEM, instead, you can take a decomposition like



having four elements with 8 12 14, and 41 nodes, respectively.

Typical functional spaces (in 3 dimensions)

Let Ω be a Lipschitz continuous polyhedral domain. The following spaces are the most common **bricks** used to deal with **PDEs**.

$L^2(\Omega)$ and $(L^2(\Omega))^3$, that we assume to be known.

$H(\mathrm{div};\Omega) := \{\boldsymbol{\tau} \in (L^2(\Omega))^3 \text{ s.t. } \mathrm{div}\,\boldsymbol{\tau} \in L^2(\Omega)\}$

$H(\mathbf{curl};\Omega) := \{\varphi \in (L^2(\Omega))^3 \text{ s.t. } \mathbf{curl}\,\varphi \in (L^2(\Omega))^3\}$

$H(\mathbf{grad};\Omega) := \{v \in L^2(\Omega) \text{ s.t. } \mathbf{grad}\,v \in (L^2(\Omega))^3\} \equiv H^1(\Omega)$

Strong formulation of Darcy's law

- p = pressure
- $\mathbf{u}$ = velocities *(volumetric flow per unit area)*
- f = source
- $\mathbb{K}$ = material-depending (full) tensor

- $\mathbf{u} = -\mathbb{K}\nabla p$ (Constitutive Equation)
- $\mathrm{div}\,\mathbf{u} = f$ (Conservation Equation)

$$-\mathrm{div}(\mathbb{K}\nabla p) = f \qquad \text{in } \Omega,$$

$$p = 0 \qquad \text{on } \partial\Omega, \qquad \text{for simplicity.}$$

We consider, as usual, the bilinear form

$$a(p, q) := \int_\Omega \mathbb{K} \nabla p \cdot \nabla q \, \mathrm{d}x$$

and we formulate the problem as: *find* $p \in H_0^1(\Omega)$ *such that:*

$$a(p, q) := \int_\Omega f \, q \mathrm{d}x \qquad \forall q \in H_0^1(\Omega).$$

Variational formulation - Mixed

The problem can also be written as: *find* $p \in L^2(\Omega)$ *and* $\mathbf{u} \in H(\mathrm{div};\Omega)$ such that

$$\int_\Omega \mathbb{K}^{-1}\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}V = \int_\Omega p\,\mathrm{div}\mathbf{v}\,\mathrm{d}V \quad \forall \mathbf{v} \in H(\mathrm{div};\Omega)$$

and

$$\int_\Omega \mathrm{div}\mathbf{u}\, q\,\mathrm{d}V = \int_\Omega f\, q\,\mathrm{d}V \qquad \forall q \in L^2(\Omega).$$

- $\mathbf{j}$ = *divergence free* current density
- μ = magnetic permeability
- $\mathbf{u}$ = vector potential with the gauge $\mathrm{div}\,\mathbf{u} = 0$
- $\mathbf{B} = \mathbf{curl}\,\mathbf{u}$ = magnetic induction
- $\mathbf{H} = \mu^{-1}\mathbf{B} = \mu^{-1}\mathbf{curl}\,\mathbf{u}$ =magnetic field
- $\mathbf{curl}\,\mathbf{H} = \mathbf{j}$

The classical magnetostatic equations become now

$$\mathbf{curl}\,\mu^{-1}\mathbf{curl}\,\mathbf{u} = \mathbf{j} \text{ in } \Omega,$$

$$\mathbf{u} \times \mathbf{n} = 0 \text{ on } \partial\Omega.$$

Variational formulation of the magnetostatic problem:

$$\begin{cases} \text{Find } \mathbf{u} \in H_0(\mathbf{curl}, \Omega) \text{ and } p \in H_0^1(\Omega) \text{ such that :} \\ (\mu^{-1}\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) - (\nabla p, \mathbf{v}) = (\mathbf{j}, \mathbf{v}) \, \forall\, \mathbf{v} \in H_0(\mathbf{curl}; \Omega) \\ \qquad\qquad\qquad\qquad (\mathbf{u}, \nabla q) = 0 \qquad \forall\, q \in H_0^1(\Omega). \end{cases}$$

For a **piecewise smooth** vector valued function, at the common boundary between two elements,

in order to belong to	**you need to match**
$(L^2(\Omega))^d$	nothing
$H(\mathrm{div};\Omega)$	normal component
$H(\mathbf{curl};\Omega)$	tangential components
$H(\mathbf{grad};\Omega)$	all the components

The following polynomial spaces are typically used, element by element, in order to approximate the above spaces:

$\mathbb{P}_0 := \{\text{constants}\}\ (1\, d.o.f.)$

$RT_0 := \{\boldsymbol{\tau} = \mathbf{a} + c\mathbf{x}\}$ with $\mathbf{a} \in \mathbb{R}^3$ and $c \in \mathbb{R}$ $(4\, d.o.f.)$

$N_0 := \{\varphi = \mathbf{a} + \mathbf{c} \wedge \mathbf{x}\}$ with $\mathbf{a} \in \mathbb{R}^3$ and $\mathbf{c} \in \mathbb{R}^3$ $(6\, d.o.f.)$

$\mathbb{P}_1 := \{v = a + \mathbf{c} \cdot \mathbf{x}\}$ with $a \in \mathbb{R}$ and $\mathbf{c} \in \mathbb{R}^3$ $(4\, d.o.f.)$

Let $\mathcal{T}_h$ be a decomposition of Ω in tetrahedra. We consider the following finite element approximations.

$$L^2(\Omega) \sim \{q \in L^2(\Omega) \text{ such that } q_{|T} \in \mathbb{P}_0 \quad \forall T \in \mathcal{T}_h\}$$

$$H(\mathrm{div};\Omega) \sim \{\boldsymbol{\tau} \in H(\mathrm{div};\Omega) \text{ s.t. } \boldsymbol{\tau}_{|T} \in RT_0 \quad \forall T \in \mathcal{T}_h\}$$

$$H(\mathbf{curl};\Omega) \sim \{\boldsymbol{\varphi} \in H(\mathbf{curl};\Omega) \text{ s.t. } \boldsymbol{\varphi}_{|T} \in N_0 \quad \forall T \in \mathcal{T}_h\}$$

$$H(\mathbf{grad};\Omega) \sim \{v \in H(\mathbf{grad};\Omega) \text{ s.t. } v_{|T} \in \mathbb{P}_1 \quad \forall T \in \mathcal{T}_h\}$$

Loss of beauty of FEM

Polynomial spaces for edge elements of degree k on cubes

$$
\begin{aligned}
span\{yz(w_2(x,z) - w_3(x,y)),\\
zx(w_3(x,y) - w_1(y,z)),\\
xy(w_1(y,z) - w_2(x,z))\}\\
+ (\mathbb{P}_k)^3 + \mathbf{grad}\, s(x,y,z)
\end{aligned}
$$

where each w_i $(i = 1, 2, 3)$ ranges over all polynomials (of 2 variables) of degree $\leq k$ and s ranges over all polynomials of *superlinear degree* $\leq k + 1$.

N.B. Superlinear degree: "ordinary degree ignoring variables that appear linearly"

What do we have have in mind

We want to use decompositions in polygons or polyhedra.

As for other methods on polyhedral elements, we will accept the trial and test functions inside each element to be rather complicated (e.g. solutions of suitable PDE's or systems of PDE's).

Contrary to other methods on polyhedral elements,

- we **will not** require the approximate evaluation of trial and test functions at the integration points.
- If possible, we would like to satisfy the *patch test*.

We consider a continuous problem; for instance
Find $u \in V \equiv H^1_0(\Omega)$ *such that*

$$a(u,v) \equiv \int_\Omega \nabla u \cdot \nabla v \, d\Omega = \int_\Omega f \, v \, d\Omega \quad \forall\, v \in V,$$

and we want to construct a discretized version: *Find* $u_h \in V_h$ *such that*

$$a_h(u_h, v_h) = (f_h\,, v_h) \quad \forall\, v_h \in V_h \subset V\,.$$

Hence we look for *sufficient conditions* on a_h and V_h that ensure *all the good properties that you would have with standard Finite Elements*.

H1 $\forall E$ $\exists$ a set of polynomials $P_E \subset V_h^E \equiv V_{h|E}$ such that

$$a_h^E(p,v) = a^E(p,v) \quad \forall v \in V^E, \forall p \in P_E$$

H2 $\exists\, \alpha^*,\ \alpha_* > 0$ such that $\forall E,\ \forall\, v \in V_h^E$:

$$\alpha_*\, a^E(v,v) \ \le\ a_h^E(v,v) \ \le\ \alpha^*\, a^E(v,v).$$

Under Assumptions **H1** and **H2** the discrete problem has a unique solution u_h, and: $\forall\, u_I \in V_h, \forall u_\pi \in \prod P_E$

$$\|u - u_h\|_1 \le C \left(\|u - u_I\|_1 + \|u - u_\pi\|_{1,h} + \|f - f_h\|_{V_h'} \right)$$

with C independent of h. And we have the *Patch Test*...

We **assume** that we know how to compute $a^E(u,v)$

whenever one of the two entries is a polynomial in P_E.

Hence, for every element E and for every $v \in V_h^E$ we can compute its *projection* $\Pi^a v \in P_E$ defined by

$$a^E(v - \Pi^a v, q) = 0 \;\forall q \in P_E, \text{ and } \pi_{K_E^a}(v - \Pi^a v) = 0.$$

where K_E^a is the kernel of a^E. Note that $\Pi^a p = p$ for all p in P_E. Then we set, for all u and v in V_h^E

$$a_h^E(u,v) := a^E(\Pi^a u, \Pi_v^a) + S(u - \Pi^a u, v - \Pi^a v)$$

where the *stabilizing* bilinear form S is (for instance) the Euclidean inner product in $\mathbb{R}^n$ (where n is the dimension of V_h^E).

Structure of the Local Matrix in a different basis

	Polynomials	Others
Polynomials	$a=a_h$	$a=a_h$
Others	$a=a_h$	$\forall$

The main features of VEM's are:

As for other methods on polyhedral elements, the trial and test functions inside each element are rather complicated (e.g. solutions of suitable PDE's or systems of PDE's).

Contrary to other methods on polyhedral elements,

- they **do not** require the approximate evaluation of trial and test functions at the integration points.
- In most cases they satisfy the *patch test* **exactly** (up to the computer accuracy).

Moreover:

- We have now a full family of spaces (for the approximation of the main functional spaces)

We take, for every integer $k \geq 1$

$$V_h^E = \{v |\; v_{|e} \in \mathbb{P}_k(e)\, \forall \text{ edge } e \text{ and } \Delta v \in \mathbb{P}_{k-2}(E)\}$$

It is easy to see that the local space will contain all $\mathbb{P}_k$.

As degrees of freedom we take:

i) the values of v at the vertices,

ii) the moments $\int_e v\, p_{k-2} \mathrm{d}e$ on each edge,

iii) the moments $\int_E v\, p_{k-2} \mathrm{d}E$ inside.

It is easy to see that these d.o.f. are *unisolvent*.

Then for every $v \in V_h^E$ and for every $p_k \in \mathbb{P}_k$

$$a^E(p_k, v) = \int_E \nabla p_k \cdot \nabla v \, dE = \int_{\partial E} \frac{\partial p_k}{\partial n} v \, d\ell - \int_E v\, \Delta p_k \, dE$$

and we see that the contribution is computable .

In every element, to *define* the trial/test function v you start from the boundary degrees of freedom, and use a 1D edge-by edge reconstruction to define the function on the whole boundary. Then you show existence and uniqueness of the reconstruction inside, using the *internal moments*.

On the other hand, to *compute* the local stiffness matrix, you use the boundary values and the internal moments to compute $a^E(v, p_k)$ for all polynomials p_k. Then you compute the operator Π^a and use it to compute a_h^E:

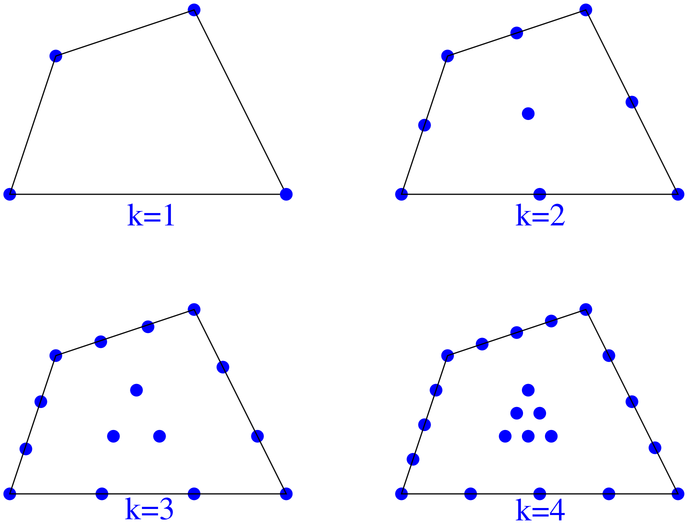
$$a_h^E(u, v) := a^E(\Pi^a u, \Pi^a v) + S(u - \Pi^a u, v - \Pi^a v).$$

The L^2-projection

A fantastic trick (sometimes called *The Three Card Monte trick*), allows the *exact* computation of the moments of order $k-1$ and k of every $v \in V_h^E$.

This is very useful for dealing with the 3D case.

Example: Degrees of freedom of nodal VEM's in 2D



For a given integer $k \geq 1$, and for every element E, we set

$$V_h^E = \{v \in H^1(E) | \, v_{|e} \in \mathbb{P}_k(e) \; \forall \text{ edge } e, \; v_{|f} \in V_h^f \; \forall \text{ face } f, \text{ and } \Delta v \in \mathbb{P}_{k-2}(E) \}$$

with the degrees of freedom:

i) values of v at the vertices,

ii) moments $\int_e v \, p_{k-2}(e)$ on each edge e,

iii) moments $\int_f v \, p_{k-2}(f)$ on each face f, and

iv) moments $\int_E v \, p_{k-2}(E)$ on E.

Ex: for $k = 3$ the number of degrees of freedom would be: the number of vertices, plus $2\times$ the number of edges, $3\times$ the number of faces, plus 4. On a cube this makes $8 + 24 + 18 + 4 = 54$ against 64 for $\mathbb{Q}_3$.

Use of the degrees of freedom in 3d

For every $v \in V_h^E$ and for every polynomial p_k of degree k

$$a^E(p_k, v) = \int_E \nabla p_k \cdot \nabla v \, \mathrm{d}E = \int_{\partial E} \frac{\partial p_k}{\partial n} v \, \mathrm{d}S - \int_E v \, \Delta p_k \, \mathrm{d}E$$

$$= \sum_{f \in \partial E} \int_f \frac{\partial p_k}{\partial n} v \, \mathrm{d}f - \int_E v \, \Delta p_k \, \mathrm{d}E$$

The term $\int_E v \, \Delta p_k \, dE$ is easy. Indeed we have that $\Delta p_k \in \mathbb{P}_{k-2}$ allowing a direct use of the degrees of freedom of v. On the contrary, on each face f we have that $\frac{\partial p_k}{\partial n}$ is in $\mathbb{P}_{k-1}$ and we need the *Three Card Monte* trick to upgrade the moments on f from $k-2$ to $k-1$

VEM approximations of $H(\mathrm{div};\Omega)$

In each element E, and for each integer k, we define

$$\mathcal{B}_k(\partial E) := \{g \,|\, g_{|e} \in \mathbb{P}_k \,\forall \text{ edge } e \in \partial E\} \text{ in 2d}$$

$$\mathcal{B}_k(\partial E) := \{g \,|\, g_{|f} \in \mathbb{P}_k \,\forall \text{ face } f \in \partial E\} \text{ in 3d}$$

$$V_k(E) = \{\boldsymbol{\tau} | \boldsymbol{\tau} \cdot \mathbf{n} \in \mathcal{B}_k(\partial E), \mathrm{div}\boldsymbol{\tau} \in \mathbb{P}_{k-1}, \mathrm{rot}\boldsymbol{\tau} \in \mathbb{P}_{k-1}\}$$

and in 3 dimensions

$$V_k(E) = \{\boldsymbol{\tau} | \boldsymbol{\tau}\cdot\mathbf{n} \in \mathcal{B}_k(\partial E), \mathrm{div}\boldsymbol{\tau} \in \mathbb{P}_{k-1}, \mathbf{curl}\boldsymbol{\tau} \in (\mathbb{P}_{k-1})^3\}$$

- $\int_e \boldsymbol{\tau} \cdot \mathbf{n}\, q_k \mathrm{d}e \qquad \forall q_k \in \mathbb{P}_k(e) \;\; \forall$ edge e
- $\int_E \boldsymbol{\tau} \cdot \mathbf{grad} q_{k-1} \mathrm{d}E \qquad \forall q_{k-1} \in \mathbb{P}_{k-1}$
- $\int_E \boldsymbol{\tau} \cdot \mathbf{g}_k^{\perp} \mathrm{d}E \qquad \forall \mathbf{g}_k^{\perp} \in \mathcal{G}_k^{\perp}$

where $\mathcal{G}_k^{\perp}$ is the subset of the $\mathbf{g} \in (\mathbb{P}_k(E))^3$ such that

$$\int_E \mathbf{g} \cdot \mathbf{grad} q_{k+1} \mathrm{d}E = 0 \quad \forall q_{k+1} \in \mathbb{P}_{k+1}(E)$$

Degrees of freedom in $V_k(E)$ in $3d$

- $\int_f \boldsymbol{\tau} \cdot \mathbf{n}\, q_k \mathrm{d}f \qquad \forall q_k \in \mathbb{P}_k(f) \;\; \forall \text{ face } f$
- $\int_E \boldsymbol{\tau} \cdot \mathbf{grad} q_{k-1} \mathrm{d}E \qquad \forall q_{k-1} \in \mathbb{P}_{k-1}$
- $\int_E \boldsymbol{\tau} \cdot \mathbf{g}_k^{\perp} \mathrm{d}E \qquad \forall \mathbf{g}_k^{\perp} \in \mathcal{G}_k^{\perp}$

  where $\mathcal{G}_k^{\perp}$ is the subset of the $\mathbf{g} \in (\mathbb{P}_k(E))^3$ such that

$$\int_E \mathbf{g} \cdot \mathbf{grad} q_{k+1} \mathrm{d}E = 0 \quad \forall q_{k+1} \in \mathbb{P}_{k+1}(E)$$

VEM approximations of $H(\mathrm{rot};\Omega)$ in $2d$

In each element E, and for each integer k, we recall

$$\mathcal{B}_k(\partial E) := \{g |\ g_{|e} \in \mathbb{P}_k \,\forall \text{ edge } e \in \partial E\} \text{ in 2d}$$

Then we set

$$V_k(E) = \{\varphi | \varphi \cdot \mathbf{t} \in \mathcal{B}_k(\partial E), \mathrm{div}\varphi \in \mathbb{P}_{k-1}, \mathrm{rot}\varphi \in \mathbb{P}_{k-1}\}$$

Degrees of freedom in $V_k(E)$ in $2d$

- $\int_e \varphi \cdot \mathbf{t}\, q_k \mathrm{d}e \qquad \forall q_k \in \mathbb{P}_k(e) \;\; \forall \text{ edge } e$
- $\int_E \varphi \cdot \mathbf{rot} q_{k-1} \mathrm{d}E \qquad \forall q_{k-1} \in \mathbb{P}_{k-1}$
- $\int_E \varphi \cdot \mathbf{r}_k^{\perp} \mathrm{d}E \qquad \forall \mathbf{r}_k^{\perp} \in \mathcal{R}_k^{\perp}$

where $\mathcal{R}_k^{\perp}$ is the subset of the $\mathbf{r} \in (\mathbb{P}_k(E))^3$ such that

$$\int_E \mathbf{r} \cdot \mathbf{rot} q_{k+1} \mathrm{d}E = 0 \quad \forall q_{k+1} \in \mathbb{P}_{k+1}(E)$$

VEM approximations of $H(\mathbf{curl};\Omega)$ in $3d$

In each element E, and for each integer k, we set

$$\mathcal{B}_k(\partial E) := \{\varphi |\ \varphi_{|f} \in V_k(f)\, \forall \text{ face } f \in \partial E \text{ and } \\ \varphi \cdot \mathbf{t}_e \text{ is single valued at each edge } e \in \partial E\}$$

Then we set

$$V_k(E) = \{\varphi| \text{ such that } \varphi \cdot \mathbf{t} \in \mathcal{B}_k(\partial E), \\ \mathrm{div}\varphi \in \mathbb{P}_{k-1}, \mathbf{curlcurl}\varphi \in (\mathbb{P}_{k-1})^3\}$$

Degrees of freedom in $V_k(E)$ in $3d$

- for every edge e $\int_e \varphi \cdot \mathbf{t}\, q_k \mathrm{d}e \quad \forall q_k \in \mathbb{P}_k(e)$
- for every face f

$$\int_f \varphi \cdot \mathbf{rot} q_{k-1} \mathrm{d}f \qquad \forall q_{k-1} \in \mathbb{P}_{k-1}(f)$$

$$\int_f \varphi \cdot \mathbf{r}_k^{\perp} \mathrm{d}f \qquad \forall \mathbf{r}_k^{\perp} \in \mathcal{R}_k^{\perp}(f)$$

- and inside E

$$\int_E \varphi \cdot \mathbf{curl} q_{k-1} \mathrm{d}E \qquad \forall q_{k-1} \in (\mathbb{P}_{k-1}(E))^3$$

$$\int_E \varphi \cdot \mathbf{r}_k^{\perp} \mathrm{d}E \qquad \forall \mathbf{r}_k^{\perp} \in \mathcal{R}_k^{\perp}(E)$$

We observe that the classical differential operators *grad*, *curl*, and *div* send these VEM spaces one into the other (up to the obvious adjustments for the polynomial degree). Indeed:

$$\mathbf{grad}(\mathit{VEM}, \mathit{nodal}) \subseteq \mathit{VEM}, \mathit{edge}$$

$$\mathbf{curl}(\mathit{VEM}, \mathit{edge}) \subseteq \mathit{VEM}, \mathit{face}$$

$$\mathrm{div}(\mathit{VEM}, \mathit{face}) \subseteq \mathit{VEM}, \mathit{volume}$$

where

$\mathit{VEM}, \mathit{volume}$ = piecewise polynomials, discontinuous.

The crucial feature

The crucial feature common to all these choices is the possibility to construct (starting from the degrees of freedom, and without solving approximate problems in the element) a symmetric bilinear form $[\mathbf{u}, \mathbf{v}]_h$ such that, on each element E, we have

$$[\mathbf{p}_k, \mathbf{v}]_h^E = \int_E \mathbf{p}_k \cdot \mathbf{v} \, \mathrm{d}E \;\; \forall \mathbf{p}_k \in (\mathbb{P}_k(E))^d, \; \forall \mathbf{v} \text{ in the VEM space}$$

and $\exists \, \alpha^* \geq \alpha_* > 0$ independent of h such that

$$\alpha_* \|\mathbf{v}\|^2_{L^2(E)} \leq [\mathbf{v}, \mathbf{v}]_h^E \leq \alpha^* \|\mathbf{v}\|^2_{L^2(E)}, \quad \forall \mathbf{v} \text{ in the VEM space}$$

The crucial feature - 2

In other words: In each VEM space (nodal, edge, face, volume) we have a corresponding inner product

$$\left[\cdot,\cdot\right]_{VEM,nodal}, \left[\cdot,\cdot\right]_{VEM,edge}, \left[\cdot,\cdot\right]_{VEM,face}, \left[\cdot,\cdot\right]_{VEM,volume}$$

that reproduces exactly the L^2 inner product whenever at least one of the two entries is a polynomial of degree $\leq k$.

Remember that

$$a(p,q) := \int_\Omega \mathbb{K}\nabla p \cdot \nabla q \,\mathrm{d}x.$$

Then we can formulate the approximate problem as:
find $p_h \in$ *VEM,nodal such that:*

$$[\mathbb{K}\,\mathbf{grad}\,p_h, \mathbf{grad}\,q_h]_{VEM,edge} = [f,\ q_h]_{VEM,nodal}$$

for all $q_h \in$ VEM,nodal.

The approximate mixed formulation can be written as:
find $p_h \in VEM, volume$ *and* $\mathbf{u}_h \in VEM, face$ such that

$$[\mathbb{K}^{-1}\mathbf{u}_h, \mathbf{v}_h]_{VEM,face} = [p_h, \mathrm{div}\mathbf{v}_h]_{VEM,volume}$$

for all $\mathbf{v}_h \in VEM, face$, and

$$[\mathrm{div}\mathbf{u}_h, q_h]_{VEM,volume} = [f,\ q_h]_{VEM,volume}$$

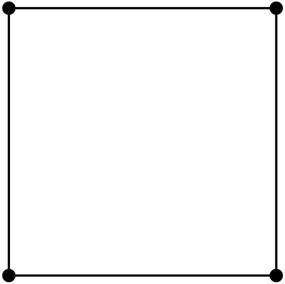
for all $q_h \in VEM, volume$.

The VEM approximation of the magnetostatic problem can be chosen as: Find $\mathbf{u}_h \in VEM, edges$ and $p_h \in VEM, nodal$ such that:

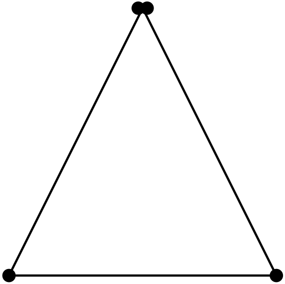
$$[\mu^{-1}\mathbf{curl}\,\mathbf{u}_h, \mathbf{curl}\,\mathbf{v}_h]_{VEM,face} - [\nabla p_h, \mathbf{v}_h]_{VEM,edge} = [\mathbf{j}, \mathbf{v}_h]_{VEM,edge}\ \forall\, \mathbf{v}_h \in VEM, edge$$

$$[\mathbf{u}, \nabla q_h]_{VEM,edge} = 0 \quad \forall\, q_h \in VEM, nodal.$$

To measure the effects of *distortion* of quadrilaterals, we solve $-\Delta u + u = f$ on the unit square, with increasingly distorted grids. The exact solution is always $ue(x, y) = \sin(2x + 0.5) * \cos(y + 0.3) + \log(1 + xy)$.

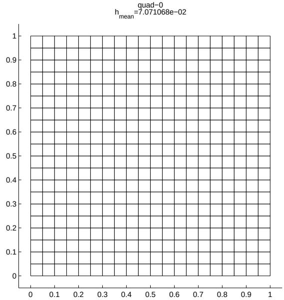


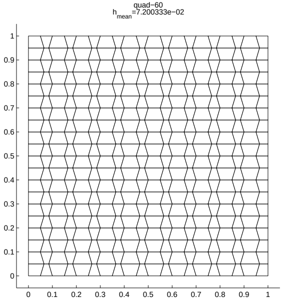
Distortion = 0%

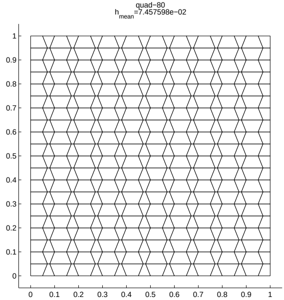


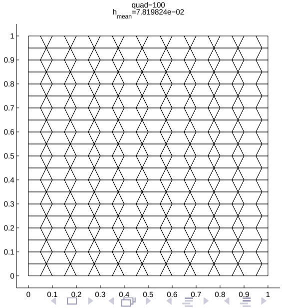
Distortion = 100%

Distortion factors 0, 60, 80, 100. MESHES

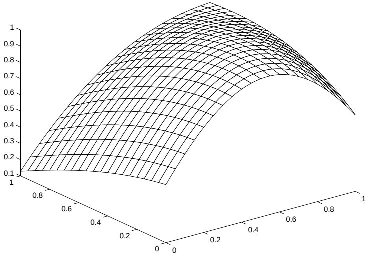


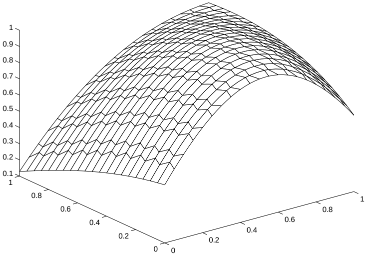


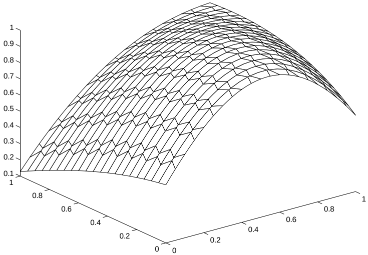


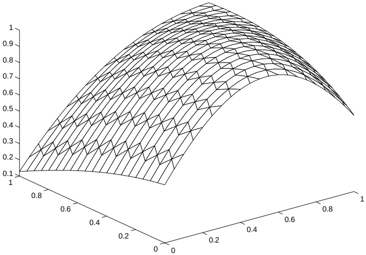


Distortion factors 0, 60, 80, 100. SOLUTIONS







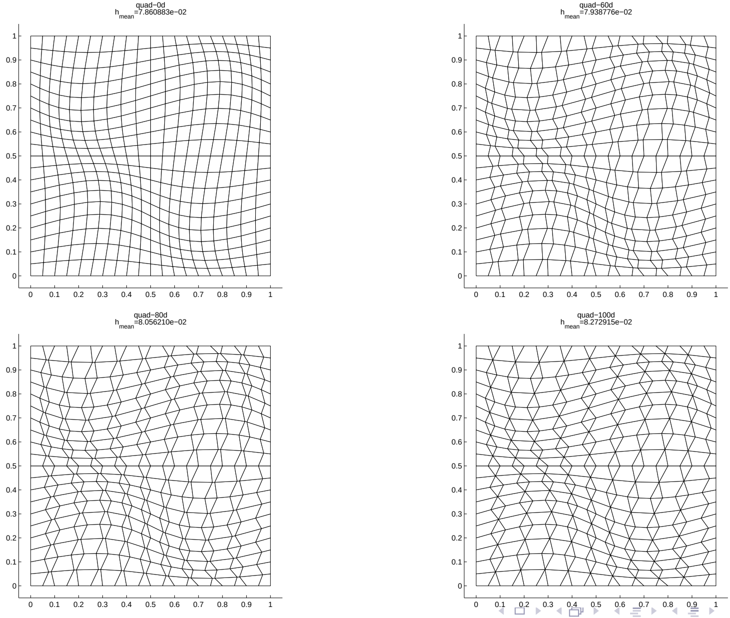


Errors for various distortion factors

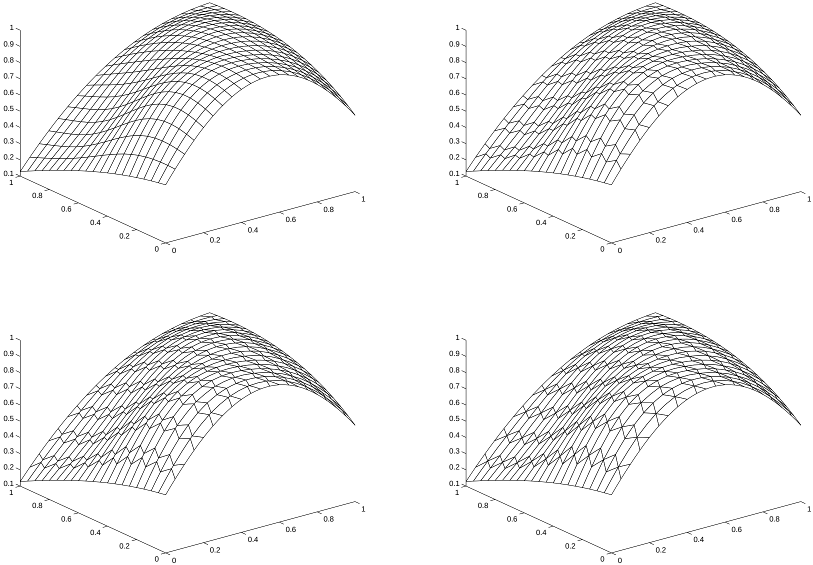
Starting from a uniform grid

Distortion	Average h	Error in l^2
0%	7.0711e-02	1.0437e-07
50%	7.1680e-02	1.6338e-06
60%	7.2003e-02	2.0469e-06
70%	7.2821e-02	2.5287e-06
80%	7.4576e-02	3.1152e-06
90%	7.6368e-02	3.8700e-06
99%	7.8013e-02	4.7784e-06
100%	7.8198e-02	4.8968e-06

Distortion factors 0, 60, 80, 100. MESHES



Distortion factors 0, 60, 80, 100. SOLUTIONS

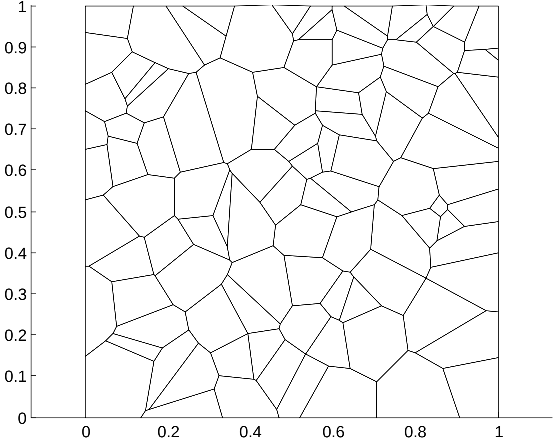


Errors for various distortion factors

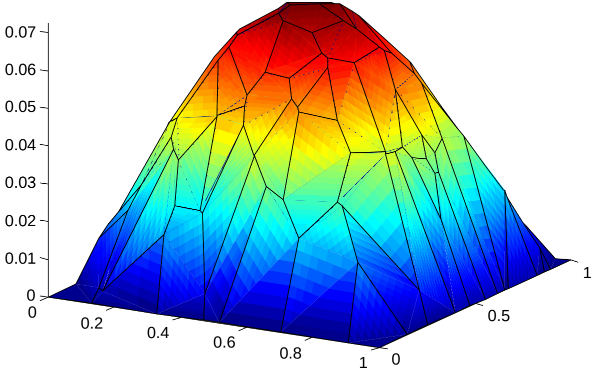
Starting from a distorted grid

Distortion	Average h	Error in l^2
0%	7.8609e-02	4.1678e-07
50%	7.9075e-02	2.3726e-06
60%	7.9388e-02	3.0113e-06
70%	7.9858e-02	3.7681e-06
80%	8.0562e-02	4.6884e-06
90%	8.1537e-02	5.8525e-06
99%	8.2599e-02	7.2161e-06
100%	8.2729e-02	7.3908e-06

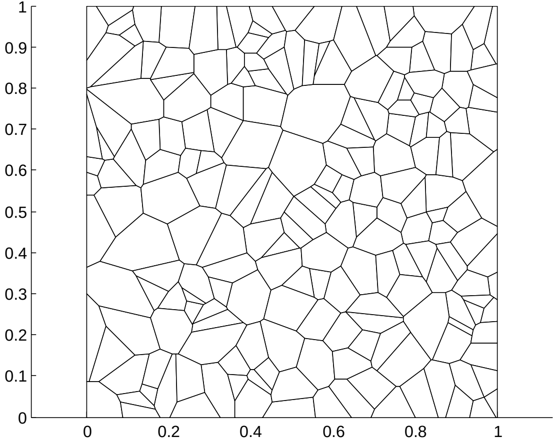
Voronoi Meshes



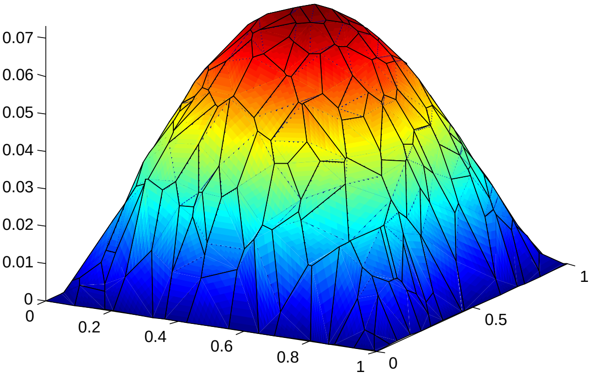
Solution



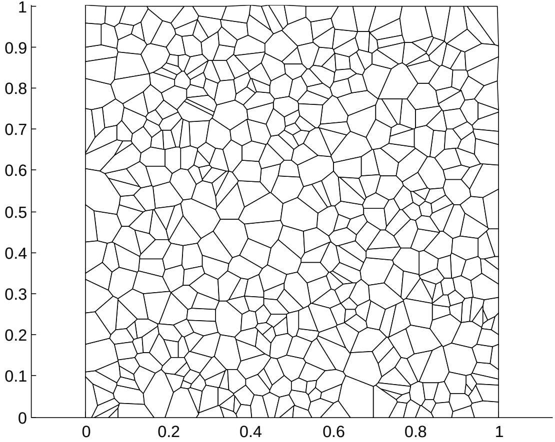
Voronoi Meshes



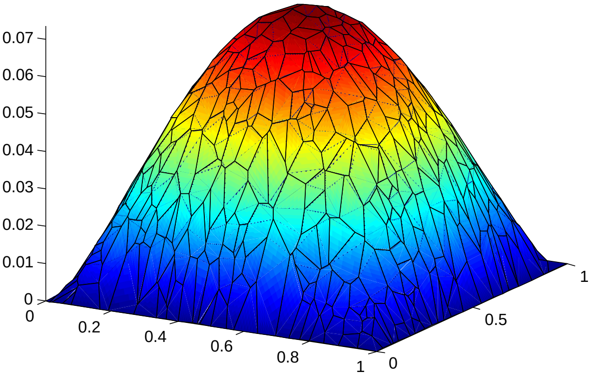
Solution



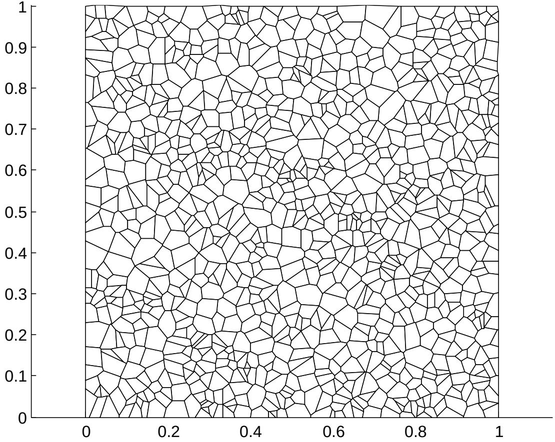
Voronoi Meshes



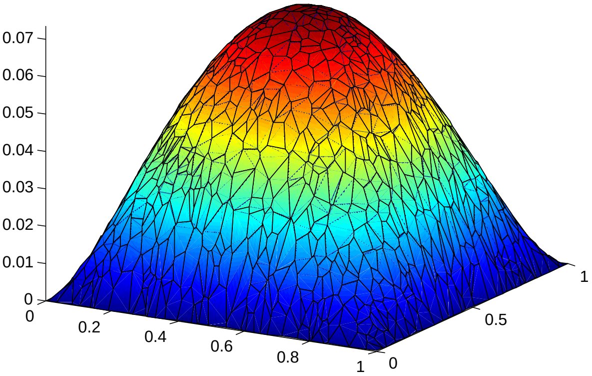
Solution



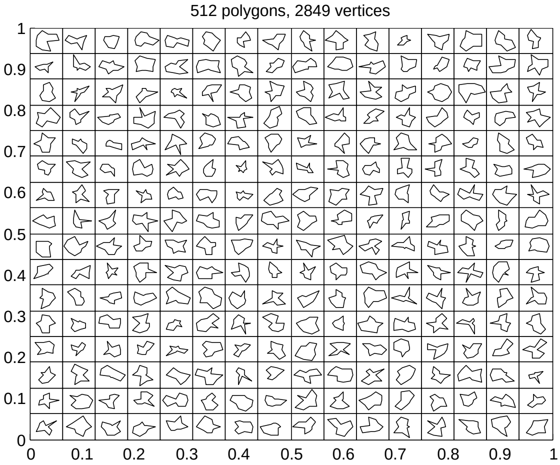
Voronoi Meshes

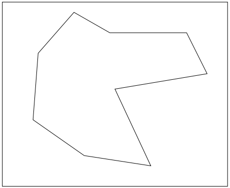


Solution

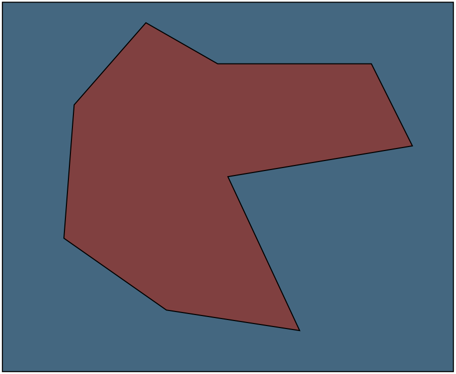


More crazy meshes



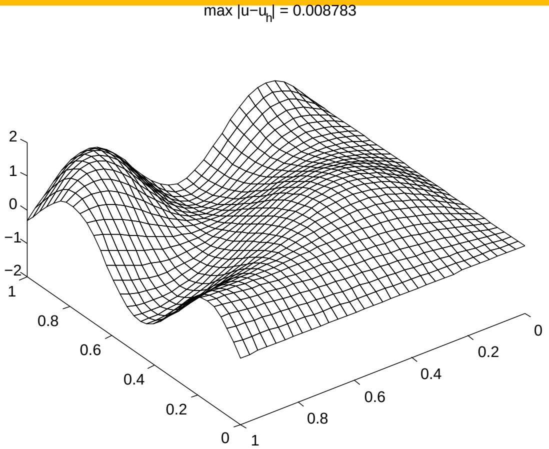


More crazy meshes



Note that the pink element is a polygon with 9 edges, while the blue element is a polygon (not simply connected) with 13 edges. We are exact on linears...

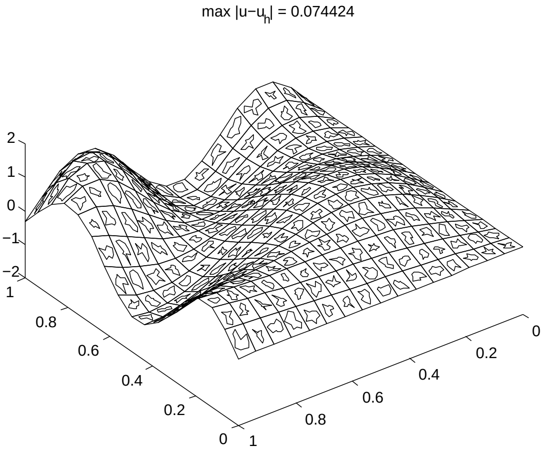
More crazy meshes



For reasons of "glastnost", we take as exact solution

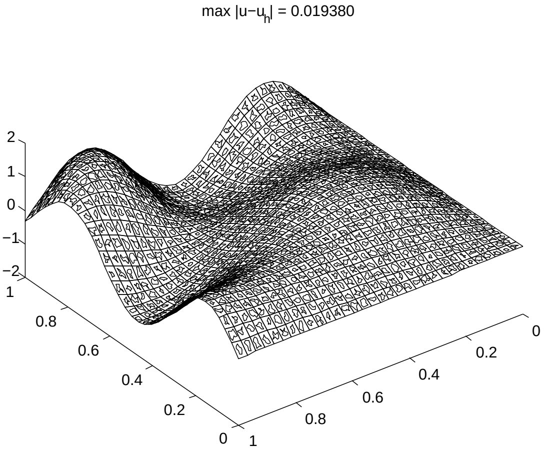
$$w = x(x-0.3)^3(2-y)^2\sin(2\pi x)\sin(2\pi y) + \sin(10xy)$$

More crazy meshes



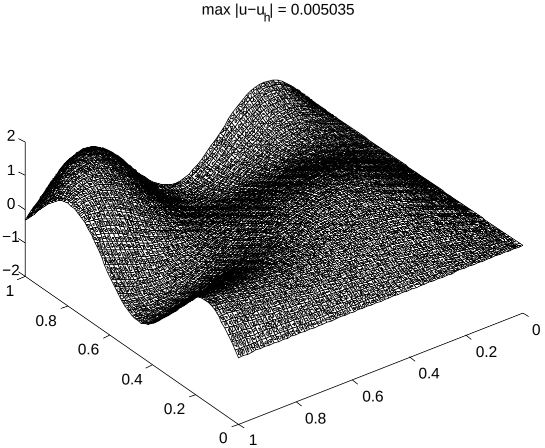
This is on a mesh of 512 (16×16 little squares) elements.

More crazy meshes



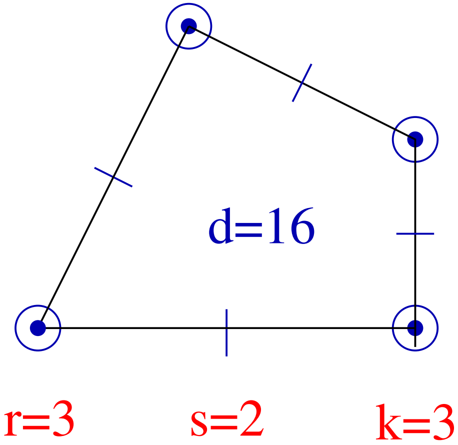
This is on a mesh of 2048 (32 $\times$ 32 little squares) elements.

More crazy meshes

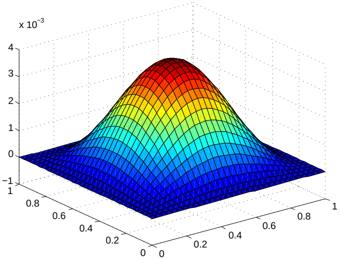


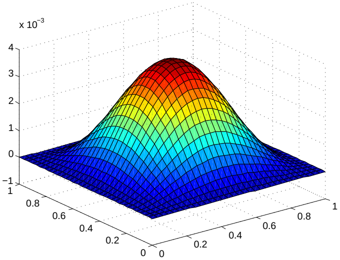
And this is on a mesh of 8192 (64×64 little squares) elements. Note the $O(h^2)$ convergence in L^∞.

The 3-2 element for plates



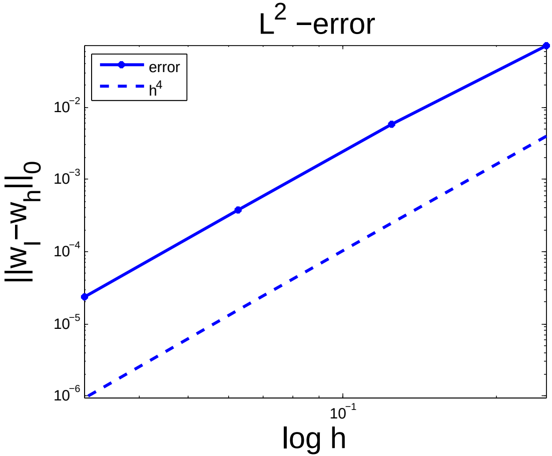
Exact and approximate solution



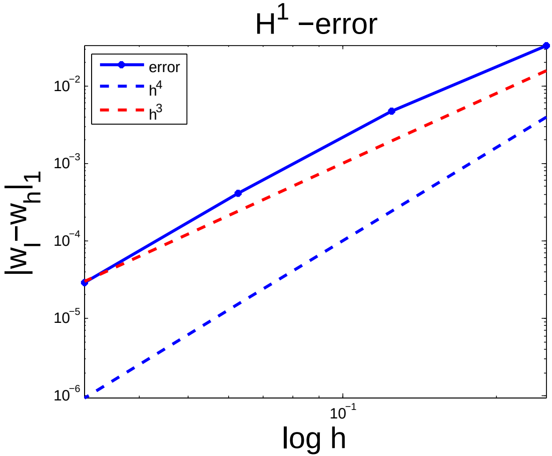


Exact solution (left): $w = x^2(1-x)^2y^2(1-y)^2$ on the unit square $]0,1[\times]0,1[$. The approximate solution is computed with the $r = 3$, $s = 2$, $k = 3$ element on a grid of uniform 32×32 square (BLUSH!) elements.

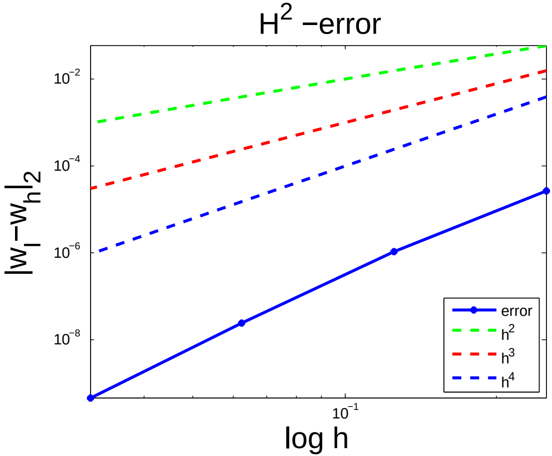
Behaviour of the L^2 error



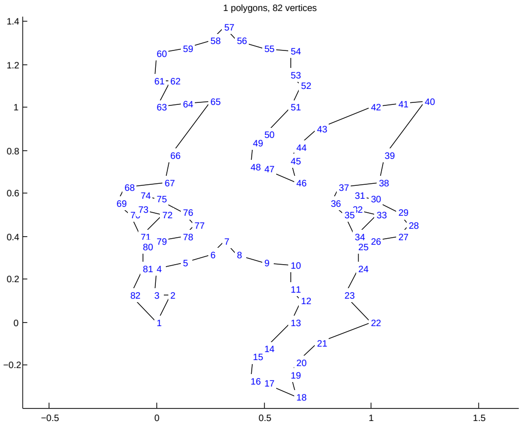
Behaviour of the H^1 error



Behaviour of the H^2 error

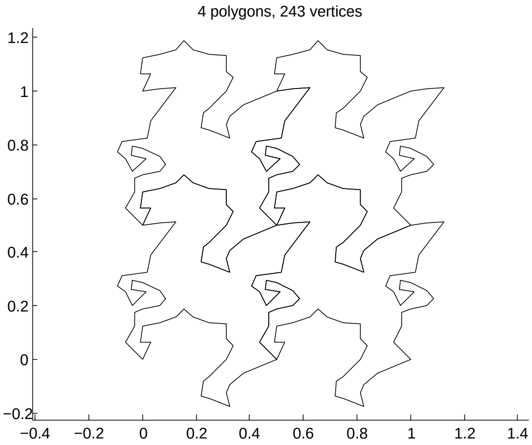


Going berserk?



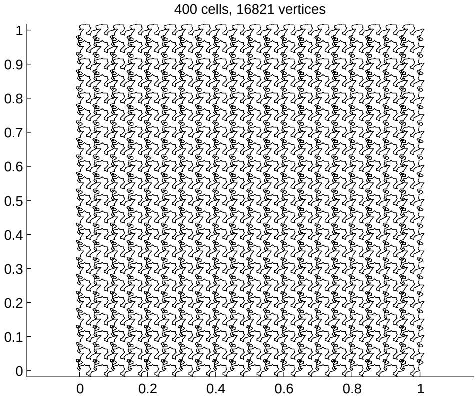
Could one use an **element** like this one, with 82 vertices?

Going berserk ?



And possibly make a mesh out of it ??.

Going totally berserk ??



And solve PDE's on a grid like this?

Come at tomorrow's lecture, and see....

Conclusions

- Virtual Elements are a new method, and a lot of work is needed to assess their *pros* and *cons*.
- Their major interest is on polygonal and polyhedral elements, but their use on distorted quads, hexa, and the like, is also quite promising.
- For triangles and tetrahedra the interest seems to be concentrated in higher order continuity (e.g. plates).
- The use of VEM mixed methods seems to be quite interesting, in particular for their connections with Finite Volumes and Mimetic Finite Differences.