Basic Principles of Mixed Virtual Element Methods

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joint work with Franco Brezzi and Rick Falk

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A brand new method

- L. Beirão da Veiga, F. Brezzi, A. Cangiani, G. Manzini, L.D. Marini, A. Russo: Basic principles of Virtual Element Methods, Math. Models Methods Appl. Sci. 23 (2013), 199-214.
- F. Brezzi, L.D. Marini: Virtual Element Method for plate bending problems, Comput. Methods Appl.Mech. Engrg. 253 (2013), 455-462.
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- B. Ahmad, A. Alsaedi, F. Brezzi, L.D. Marini, A. Russo: Equivalent projectors for virtual element methods, Comput. Math. Appl. 66 (2013), 376-391.
- L. Beirão da Veiga, G. Manzini: A virtual element method with arbitrary regularity, IMA J. Numer. Anal. (2013) doi: 10.1093/imanum/drt018.
- L. Beirão da Veiga, F. Brezzi, L.D. Marini, A. Russo: Hitchhikers Guide to the Virtual Element Method, Math. Models Methods Appl. Sci.(2014) doi: 10.1142/S021820251440003X.
- A. Cangiani, G. Manzini, A. Russo, N. Sukumar: Hourglass stabilization and the virtual element method (submitted)

VEM vs. FEM in few words

- Similarities:
  same starting point, i.e., variational formulation of the given problem;
  spaces of polynomials of a given degree are included.
- Differences:
  grids made of polygons of arbitrary shape can be used;
  easy to construct high-regularity approximations.

Polygonal and polyhedral grids

There is a wide literature on Polygonal and Polyhedral Elements

- Rational Polynomials (Wachspress, 1975, 2010)
- Voronoi tassellations (Sibson, 1980; Hiyoshi-Sugihara, 1999; Sukumar et als, 2001)
- Mean Value Coord. (Floater, 2003)
- Metric Coord. (Malsch-Lin-Dasgupta, 2005)
- Maximum Entropy (Arroyo-Ortiz, 2006; Hormann-Sukumar, 2008)
- Harmonic Coord. (Joshi et als 2007; Martin et als, 2008; Bishop 2013)

Why Polygonal/Polyhedral Elements

There are several types of problems where Polygonal and Polyhedral elements are used:

- Crack propagation and Fractured materials (e.g. T. Belytschko, N. Sukumar)
- Topology Optimization (e.g. O. Sigmund, G.H. Paulino)
- Computer Graphics (e.g. M.S. Floater)
- Fluid-Structure Interaction (e.g. W.A. Wall)
- Complex Microstructures (e.g. N. Moes)
- Two-phase flows (e.g. J. Chessa)

Outline

1. Model problem - Reminders on Virtual Element Methods
2. Model problem - Mixed FEM and VEM
3. Numerical Results

The continuous problem

$\Omega \subset \mathbb{R}^2$ (polygonal) computational domain, $f \in L^2(\Omega)$ source term
We look for $p \in H^1(\Omega)$ (pressure) solution of

$$-\mathrm{div}(\mathbb{K}\,\mathbf{grad}\,p) = f \quad \text{in } \Omega \qquad (\mathbb{K}\,\mathbf{grad}\,p)\cdot\mathbf{n} = \mathbf{0} \text{ on } \boldsymbol{\Gamma} \equiv \partial\boldsymbol{\Omega}$$

$\mathbb{K}$ *permeability tensor*, symmetric and positive definite
(for simplicity, constant or piecewise constant).

Compatibility conditions and uniqueness:

$$\int_\Omega f \, \mathrm{d}\Omega = 0 \qquad \text{and} \qquad \int_\Omega p \, \mathrm{d}\Omega = 0.$$

Reminders on Virtual Elements

Continuous problem:

find $p \in Q := H^1(\Omega)$ such that $a(p,q) = (f,q) \qquad \forall q \in Q$

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$(a(p,q) = \int_\Omega \mathbb{K}\, \mathbf{grad}\, p \cdot \mathbf{grad}\, q \,\mathrm{d}\Omega, \quad (f,q) = \int_\Omega fq \,\mathrm{d}\Omega)$

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Continuous problem:

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We need to define:

- Q_h: a finite dimensional space ($\subset Q$ for continuous VEM)

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in such a way that the problem

find $p_h \in Q_h$ such that $a_h(p_h, q_h) = (f_h, q_h) \qquad \forall q_h \in Q_h$

has a unique solution, and optimal error estimates hold.

Approximation

$\mathcal{T}_h$ a decomposition of Ω into polygons E

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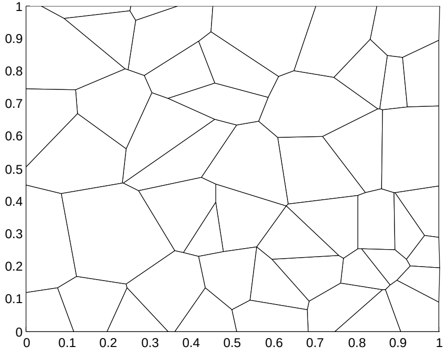

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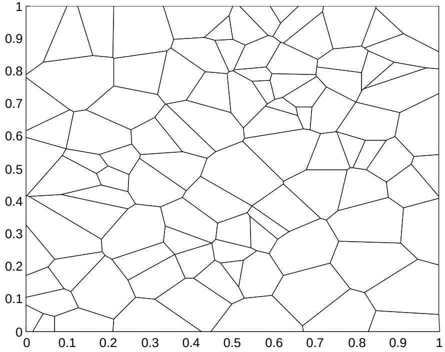

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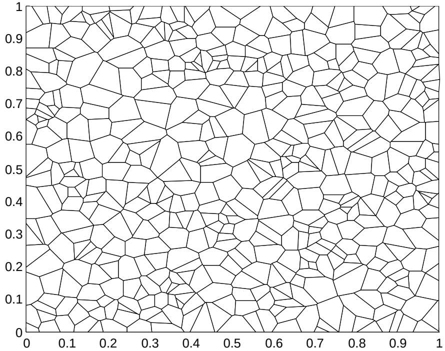

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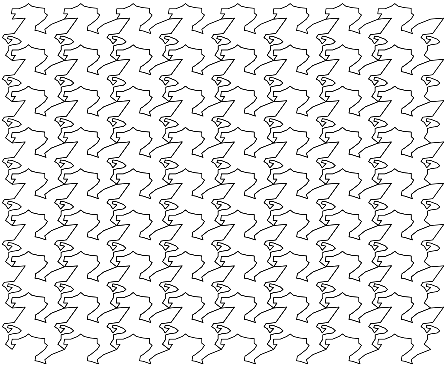

Reminders on continuous VEM-General Assumptions

We fix an integer $k \geq 1$ (our order of accuracy).

For all h, and for all E in $\mathcal{T}_h$:

H1- $\forall p_k \in \mathbb{P}_k, \forall q_h \in Q_h$

$$a_h^E(p_k, q_h) = a^E(p_k, q_h) \qquad k - Consistency$$

H2- $\exists$ two positive constants α_* and α^*, independent of h and of E, such that

$$\forall q_h \in Q_h \qquad \alpha_* \, a^E(q_h, q_h) \leq a_h^E(q_h, q_h) \leq \alpha^* \, a^E(q_h, q_h) \quad Stability$$

Convergence

Under these assumptions we have:

Theorem

The discrete problem: Find $p_h \in Q_h$ such that

$$a_h(p_h, q_h) = (f_h, q_h), \qquad \forall\, q_h \in Q_h$$

has a unique solution p_h. Moreover, for every approximation p_I of p in Q_h and for every approximation p_π of p that is piecewise in $\mathbb{P}_k$, we have

$$\|p - p_h\|_Q \le C\Big(\|p - p_I\|_Q + \|p - p_\pi\|_{h,Q} + \|f - f_h\|_{Q'}\Big)$$

where C is a constant independent of h.

Proof of convergence

Continuous VEM for the model problem

$-\mathrm{div}(\mathbb{K}\,\mathbf{grad}\,p) = f \quad in\ \Omega, \quad (\mathbb{K}\,\mathbf{grad}\,p)\cdot\mathbf{n} = 0\ on\ \partial\Omega.$ For $k \geq 1$:

Continuous VEM for the model problem

$$Q_h := \{q \in H^1(\Omega) : \; q_{|e} \in \mathbb{P}_k(e) \; \forall e \in \mathcal{T}_h, \; \mathrm{div}(\mathbb{K}\, \mathbf{grad}\, q) \in \mathbb{P}_{k-2}(E) \; \forall E\}$$

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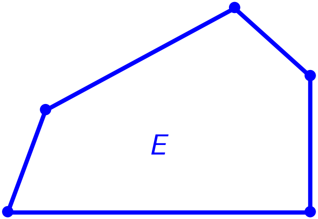
Example: $k = 1$



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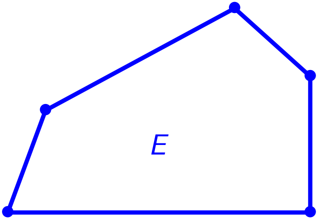

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$$a_h(p_h, q_h) \simeq a(p_h, q_h) := \int_\Omega \mathbb{K}\,\mathbf{grad}\, p_h \cdot \mathbf{grad}\, q_h d\Omega$$

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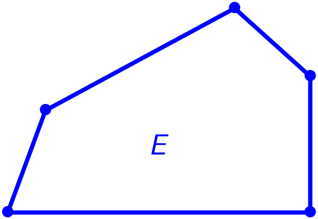

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$$a^E(p_1, q) = \int_E \mathbb{K}\,\mathbf{grad}\, p_1 \cdot \mathbf{grad}\, q \, dE = \int_{\partial E} \mathbb{K}\mathbf{grad}\, p_1 \cdot \mathbf{n} q \, d\ell =: a_h^E(p_1, q)$$

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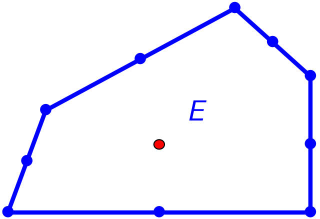

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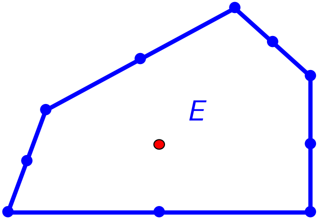

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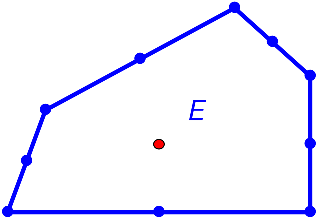

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$$a^E(p_2, q) = -\int_E \mathrm{div}(\mathbb{K}\, \mathbf{grad}\, p_2) q dE + \int_{\partial E} \mathbb{K} \mathbf{grad}\, p_2 \cdot \mathbf{n} q d\ell =: a_h^E(p_2, q)$$

How to construct a globally computable $a_h(\cdot,\cdot)$

Def: $\Pi_k^{\nabla} v \in \mathbb{P}_k(E)$

$$\begin{cases} a^E(\Pi_k^{\nabla} v, q) = a^E(v,q) \ \forall q \in \mathbb{P}_k(E) \\ \displaystyle\int_{\partial E} \Pi_k^{\nabla} v d\ell = \int_{\partial E} v d\ell \end{cases}$$

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with $\mathcal{S}(\cdot,\cdot)$ any symmetric bilinear form that scales like $a(\cdot,\cdot)$:

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Consistency and Stability

Back to the continuous problem

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Mixed formulation

$$\mathbf{u} = -\mathbb{K}\,\mathbf{grad}\, p, \qquad \mathrm{div}\mathbf{u} = f \text{ in } \Omega, \qquad \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma.$$

spaces: $V := \{\mathbf{v} \in H(\mathrm{div};\Omega) \text{ s.t. } \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma\}, \quad Q := L^2(\Omega)/\mathbb{R},$

norms: $\|\mathbf{v}\|_V^2 = \int_\Omega |\mathbf{v}|^2 \,\mathrm{d}\Omega + \int_\Omega |\mathrm{div}\mathbf{v}|^2 \,\mathrm{d}\Omega, \quad \|q\|_Q^2 = \int_\Omega |q|^2 \,\mathrm{d}\Omega,$

bilinear forms:

$$a(\mathbf{u},\mathbf{v}) := \int_\Omega \mathbb{K}^{-1}\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}\Omega \qquad b(\mathbf{v},q) := \int_\Omega \mathrm{div}\mathbf{v}\, q\,\mathrm{d}\Omega$$

Find $(\mathbf{u},p) \in V \times Q$ *such that:*

$$\begin{cases} a(\mathbf{u},\mathbf{v}) - b(\mathbf{v},p) = 0 & \forall\,\mathbf{v} \in V, \\ b(\mathbf{u},q) = (f,q) & \forall q \in Q. \end{cases}$$

Mixed Finite Elements

RT_k family on triangular grids:

$$k \geq 0: \quad Q_h := \{q \in L^2(\Omega): \; q_{|E} \in \mathbb{P}_k(E) \; \forall E \in \mathcal{T}_h\}$$

$$V_h := \{\mathbf{v} \in H(\mathrm{div}, \Omega): \; \mathbf{v}_{|E} = [\mathbb{P}_k(E)]^2 \oplus \mathbf{x}\mathbb{P}_k \; \forall E \in \mathcal{T}_h\}$$

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d.o.f. in V_h:

- $\int_e \mathbf{v} \cdot \mathbf{n} \, q \, d\ell \qquad \forall \, q \in \mathbb{P}_k(e) \qquad \forall$ edge e in $\mathcal{T}_h$,
- $\int_E \mathbf{v} \cdot \mathbf{q} \, dE \qquad \forall \, \mathbf{q} \in [\mathbb{P}_{k-1}(E)]^2 \qquad \forall E \in \mathcal{T}_h$.

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- $\int_E \mathbf{v} \cdot \mathbf{curl} b \, dE \qquad \forall\, b \in B_{k+1} \qquad \forall E \in \mathcal{T}_h$.

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- V_h: a finite dimensional space $\subset V$
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- a bilinear form $a_h(\cdot,\cdot): \; V_h \times V_h \; \rightarrow \; \mathbb{R}$
- a bilinear form $b_h(\cdot,\cdot): \; V_h \times Q_h \; \rightarrow \; \mathbb{R}$
- an element $f_h \in V_h'$

in such a way that the problem

find $(\mathbf{u}_h, p_h) \in V_h \times Q_h$ such that

$$\begin{cases} a_h(\mathbf{u}_h, \mathbf{v}_h) - b_h(\mathbf{v}_h, p_h) = 0 & \forall \, \mathbf{v}_h \in V_h, \\ b_h(\mathbf{u}_h, q_h) = (f_h, q_h) & \forall q_h \in Q_h, \end{cases}$$

has a unique solution, and optimal error estimates hold.

Choice of the spaces V_h and Q_h, BDM_k – *extension* to polygons

$k \geq 1 \longrightarrow Q_h := \{q \in Q \text{ s.t. } q_{|E} \in \mathbb{P}_{k-1}(E) \text{ for all element } E \in \mathcal{T}_h\}$

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$$V_h := \{\mathbf{v} \in V \text{ s.t. } (\mathbf{v} \cdot \mathbf{n})_{|e} \in \mathbb{P}_k(e) \text{ for all edge } e \text{ in } \mathcal{T}_h,$$
$$(\mathrm{div}\mathbf{v})_{|E} \in \mathbb{P}_{k-1}(E) \text{ and } (\mathrm{rot}\,\mathbf{v})_{|E} \in \mathbb{P}_{k-1}(E) \text{ for all element } E \in \mathcal{T}_h\}$$

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d.o.f. in V_h:

- $\displaystyle\int_e \mathbf{v}\cdot\mathbf{n}\, q\, d\ell \qquad \forall\, q \in \mathbb{P}_k(e) \qquad \forall \text{ edge } e \text{ in } \mathcal{T}_h,$
- $\displaystyle\int_E \mathbf{v}\cdot\mathbf{grad}\, q\, dE \qquad \forall\, q \in \mathbb{P}_{k-1}(E) \qquad \forall E \in \mathcal{T}_h,$
- $\displaystyle\int_E \mathrm{rot}\,\mathbf{v}\, q\, dE \qquad \forall\, q \in \mathbb{P}_{k-1}(E) \qquad \forall E \in \mathcal{T}_h.$

Unisolvence

Lemma

The d.o.f. are unisolvent

Proof of unisolvence

Interpolation in V_h and Q_h

$p \in Q \quad \longrightarrow \quad p_I \in Q_h$ as

$$\int_E (p - p_I) q_{k-1} dE = 0 \qquad \forall E \in \mathcal{T}_h, \ \forall q_{k-1} \in \mathbb{P}_{k-1}(E).$$

$p_I = P^E_{k-1} p := L^2-$projection onto $\mathbb{P}_{k-1}(E) \ \forall E \in \mathcal{T}_h$:

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$p_I = P^E_{k-1} p :=$ L^2-projection onto $\mathbb{P}_{k-1}(E)$ $\forall E \in \mathcal{T}_h$:

$$\|p - p_I\|_{0,E} \leq C\, h_E^k\, |p|_{k,E}$$

Interpolation in V_h and Q_h

Let $\mathbf{w} \in V$ (plus $\mathbf{w} \in (L^s(\Omega))^2$ for some $s > 2$, and also $\operatorname{rot} \mathbf{w} \in L^1(E)$).
Define its interpolant $\mathbf{w}_I \in V_h$ as:

- $\int_e (\mathbf{w} - \mathbf{w}_I) \cdot \mathbf{n}\, q \,\mathrm{d}\ell = 0 \qquad \forall\, q \in \mathbb{P}_k(e) \qquad \forall$ edge e in $\mathcal{T}_h$,
- $\int_E (\mathbf{w} - \mathbf{w}_I) \cdot \mathbf{grad}\, q dE = 0 \qquad \forall\, q \in \mathbb{P}_{k-1}(E) \quad \forall E \in \mathcal{T}_h$,
- $\int_E \operatorname{rot}(\mathbf{w} - \mathbf{w}_I)\, q dE = 0 \qquad \forall\, q \in \mathbb{P}_{k-1}(E) \quad \forall E \in \mathcal{T}_h$.

$\mathbf{w}_I$ exists and is unique.

Interpolation in V_h and Q_h

Moreover,

$$\int_E \mathrm{div}(\mathbf{w} - \mathbf{w}_I)\, q_{k-1} dE = -\int_E (\mathbf{w} - \mathbf{w}_I) \cdot \mathbf{grad}\, q_{k-1} dE + \int_{\partial E} (\mathbf{w} - \mathbf{w}_I) \cdot \mathbf{n}\, q_{k-1} d\ell = 0$$

Interpolation in V_h and Q_h

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Since $\mathrm{div}\, \mathbf{w}_I \in \mathbb{P}_{k-1}(E) \implies \mathrm{div}\, \mathbf{w}_I = P^E_{k-1} \mathrm{div}\, \mathbf{w}$.

Interpolation in V_h and Q_h

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Since $\mathrm{div}\, \mathbf{w}_I \in \mathbb{P}_{k-1}(E) \implies \mathrm{div}\, \mathbf{w}_I = P^E_{k-1} \mathrm{div}\, \mathbf{w}$.
Hence,

$$\|\mathrm{div}\, \mathbf{w} - \mathrm{div}\, \mathbf{w}_I\|_{0,E} \le C\, h_E^k\, |\mathrm{div}\, \mathbf{w}|_{k,E}$$

and

$$\|\mathbf{w} - \mathbf{w}_I\|_{0,E} \le C\, h_E^{k+1}\, |\mathbf{w}|_{k+1,E}$$

The discrete bilinear forms

$$b_h(\mathbf{v}, q) \equiv b(\mathbf{v}, q) := \sum_{E\in\mathcal{T}_h} \int_E \mathrm{div}\mathbf{v}\, q\, dE \qquad \mathbf{v} \in V_h,\ q \in Q_h,$$

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computable from the d.o.f.:

$$\int_E \mathrm{div}\mathbf{v}\, q\, dE = -\int_E \mathbf{v} \cdot \mathbf{grad} q\, dE + \int_{\partial E} \mathbf{v} \cdot \mathbf{n} q\, \mathrm{d}s$$

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The bilinear form a_h needs more care

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Note: on each element E, whenever $\widehat{\mathbf{u}}$ is of the form

$$\widehat{\mathbf{u}} = \mathbb{K}\,\mathbf{grad}\,\widehat{q}_{k+1} \quad \text{with } \widehat{q}_{k+1} \in \mathbb{P}_{k+1},$$

then for every $\mathbf{v} \in V_{h|E}$, we have

$$a^E(\widehat{\mathbf{u}}, \mathbf{v}) = \int_E \mathbb{K}^{-1}\widehat{\mathbf{u}} \cdot \mathbf{v}\, dE = \int_E \mathbf{grad}\,\widehat{q}_{k+1} \cdot \mathbf{v}\, dE$$
$$= -\int_E \widehat{q}_{k+1}\mathrm{div}\mathbf{v}\, dE + \int_{\partial E} \widehat{q}_{k+1}\mathbf{v} \cdot \mathbf{n}\, \mathrm{d}s =: a_h^E(\widehat{\mathbf{u}}, \mathbf{v}).$$

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(computable from the d.o.f.)

The operator $\widehat{\Pi}^E$

Define first, for each element E, the space

$$\widehat{V}^E := \{\widehat{\mathbf{v}} \in V_{h|E} \text{ such that } \widehat{\mathbf{v}} = \mathbb{K}\,\mathbf{grad}\,\widehat{q}_{k+1} \;\text{ for some } \widehat{q}_{k+1} \in \mathbb{P}_{k+1}(E)\}.$$

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Next: let $\widehat{\Pi}^E : H(\mathrm{div}; E) \longrightarrow \widehat{V}^E$ be defined by:

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($\widehat{\Pi}^E\mathbf{v}$ is computable from the d.o.f. of $\mathbf{v}$)

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Choose $a_h^E(\mathbf{u}, \mathbf{v}) := a^E(\widehat{\Pi}^E\mathbf{u}, \widehat{\Pi}^E\mathbf{v}) + \mathcal{S}^E((I - \widehat{\Pi}^E)\mathbf{u}, (I - \widehat{\Pi}^E)\mathbf{v})$

$\mathcal{S}^E(\cdot,\cdot)$ is any symmetric positive definite bilinear form that scales like $a^E(\cdot,\cdot)$:

$$c_0\, a^E(\mathbf{v}, \mathbf{v}) \le \mathcal{S}^E(\mathbf{v}, \mathbf{v}) \le c_1\, a^E(\mathbf{v}, \mathbf{v}) \;\; \forall E \in \mathcal{T}_h,\; \forall \mathbf{v} \in V^E\;.$$

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Consistency and Stability

Continuity and ellipticity on the kernel

Symmetry of a_h and stability give continuity:

$$a_h^E(\mathbf{u},\mathbf{v}) \le (a_h^E(\mathbf{u},\mathbf{u}))^{1/2}(a_h^E(\mathbf{v},\mathbf{v}))^{1/2} \le \alpha^*(a^E(\mathbf{u},\mathbf{u}))^{1/2}(a^E(\mathbf{v},\mathbf{v}))^{1/2}$$

$$\le \alpha^* \; M\|\mathbf{u}\|_{0,E}\|\mathbf{v}\|_{0,E}$$

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Stability again implies

$$a_h(\mathbf{v},\mathbf{v}) \ge \alpha_*\alpha\|\mathbf{v}\|^2_{L^2} \qquad \forall \mathbf{v} \in V_h$$

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Defining the discrete kernel

$$\mathcal{K}_h := \{\mathbf{v}_h \in V_h \text{ s. t. } b(\mathbf{v}_h, q_h) = 0 \; \forall q_h \in Q_h\} \equiv \{\mathbf{v}_h \in V_h \text{ s. t. } \operatorname{div} \mathbf{v}_h = 0\},$$

we see that

$$\mathcal{K}_h \subset \mathcal{K}$$

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we see that

$$\mathcal{K}_h \subset \mathcal{K}$$

Hence,

$$a_h(\mathbf{v},\mathbf{v}) \ge \alpha_*\alpha\|\mathbf{v}\|^2_V \qquad \forall \mathbf{v} \in \mathcal{K}_h.$$

Inf-Sup condition

We have the following results:

Theorem

There exists a constant $\beta^* > 0$ independent of h such that:

$$\forall\, q^* \in Q_h,\ \exists\, \mathbf{w}_h^* \in V_h \ \text{ such that } \operatorname{div} \mathbf{w}_h^* = q^* \ \text{ and } \beta^* \|\mathbf{w}_h^*\|_V \leq \|q^*\|_Q.$$

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Proof: from the continuous Inf-Sup:

$$\forall\, q^* \in Q_h,\ \exists \mathbf{w}^* \in [H_0^1(\Omega)]^2 \text{ such that } \operatorname{div} \mathbf{w}^* = q^* \text{ and } \beta \|\mathbf{w}^*\|_{1,\Omega} \leq \|q^*\|_Q.$$

Take the interpolant $\mathbf{w}_I^*$, that verifies

$$\operatorname{div} \mathbf{w}_I^* = P_{Q_h} \operatorname{div} \mathbf{w}^* = P_{Q_h} q^* = q^*$$

$$\|\mathbf{w}_I^*\|_0 \leq (1 + C\,h) \|\mathbf{w}^*\|_{1,\Omega} \leq \frac{1 + C\,h}{\beta} \|q^*\|_Q$$

Convergence

Theorem

The discrete problem:

$$\begin{cases} \text{Find } (\mathbf{u}_h, p_h) \text{ in } V_h \times Q_h \text{ such that} & \\ a_h(\mathbf{u_h}, \mathbf{v}) - b(\mathbf{v}, p_h) = 0 & \forall\, \mathbf{v} \in V_h, \\ b(\mathbf{u}_h, q) = (f, q) & \forall q \in Q_h \end{cases}$$

has a unique solution $(\mathbf{u}_h, p_h)$*. Moreover, for every approximation* $\mathbf{u}_\pi$ *of* $\mathbf{u}$ *that is piecewise in* $\widehat{V}^E$*:*

$$\|\mathbf{u_h} - \mathbf{u}\|_{L^2} \leq C_1\Big(\|\mathbf{u} - \mathbf{u_I}\|_{L^2} + \|\mathbf{u} - \mathbf{u}_\pi\|_{L^2}\Big),$$

$$\|p_I - p_h\|_Q \leq C_2\Big(\|\mathbf{u} - \mathbf{u_h}\|_{L^2} + \|\mathbf{u} - \mathbf{u}_\pi\|_{L^2}\Big),$$

where C_1, C_2 *are constants independent of* h*.*

Proof of convergence

Orders of convergence

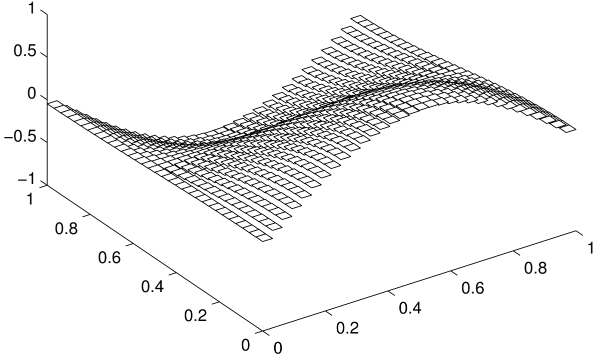
Corollary

The following estimates hold:

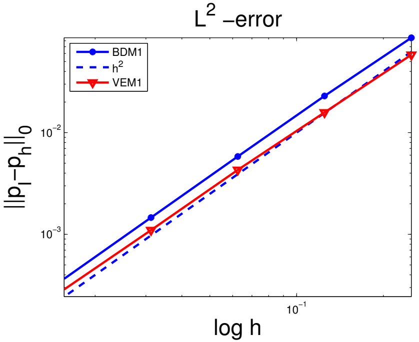
$$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega} \leq \; C\, h^{k+1}\|\mathbf{u}\|_{k+1,\Omega} \qquad \|\mathrm{div}(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega} \leq \; C\, h^{k}\|f\|_{k,\Omega},$$

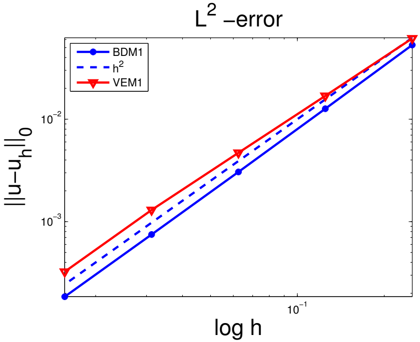
$$\|p_I-p_h\|_{0,\Omega} \leq \; C\, h^{k+1}\|\mathbf{u}\|_{k+1,\Omega} \qquad \|p-p_h\|_{0,\Omega} \leq \; C\, h^{k}(\|p\|_{k,\Omega}+\|\mathbf{u}\|_{k,\Omega}).$$

Mesh of squares: $4x4, 8x8, \ldots\ldots.., 64x64$
Exact solution: $p = \sin(2x)\cos(3y)$



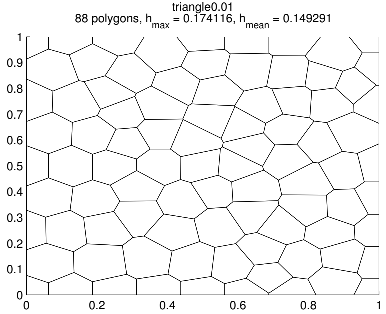
Numerical results

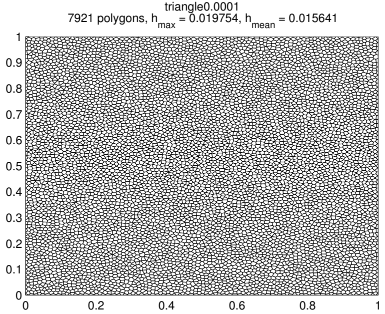




Numerical results

Voronoi polygons: 88, ..., 7921
Exact solution: $p = \sin(2x)\cos(3y)$





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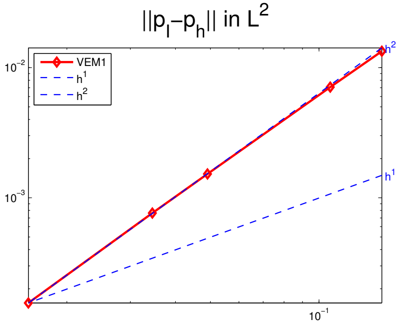
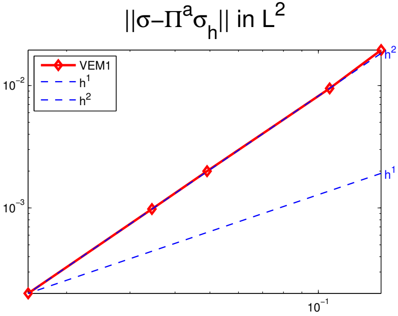
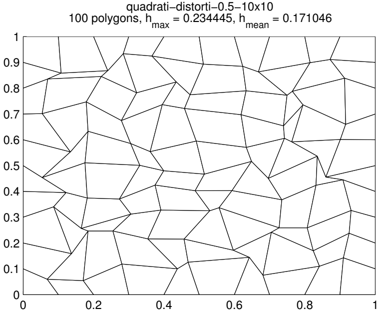
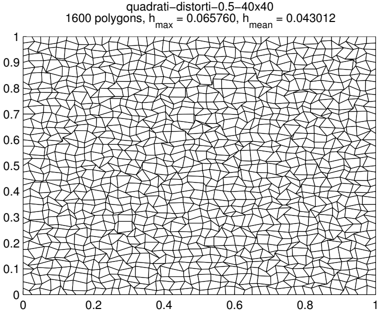




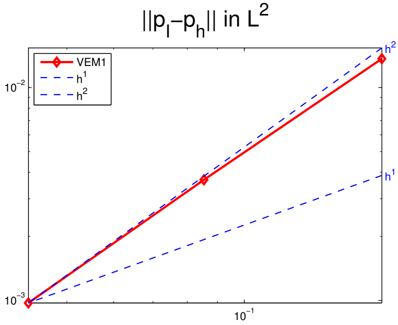
Numerical results

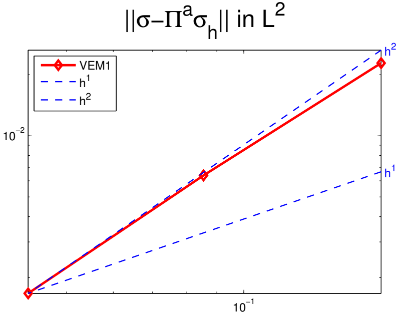
Mesh of distorted quads: $10x10, 20x20, 40x40$
Exact solution: $p = \sin(2x)\cos(3y)$





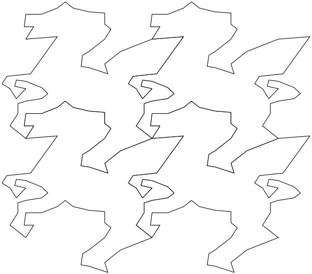
Numerical results

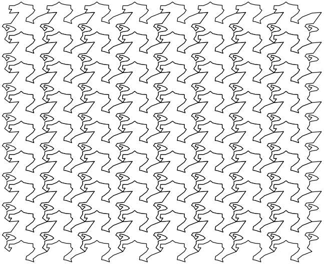




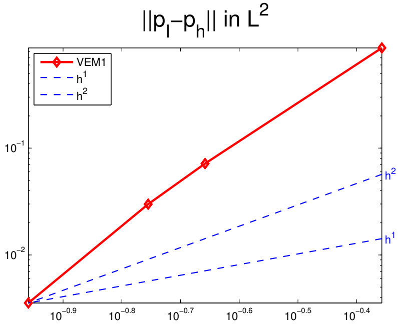
Numerical results

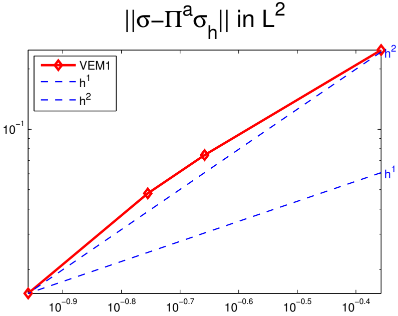
Mesh of horses: $4x4, 8x8, 10x10, 16x16$
Exact solution: $p = \sin(2x)\cos(3y)$





Numerical results





Other "crazy" grids

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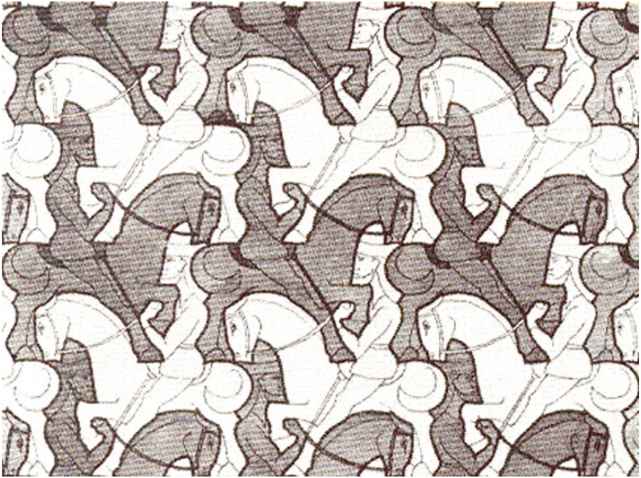

Proof of convergence

Set $\delta_h := p_h - p_I$ ◂ back0

$$\alpha_* \, \alpha \|\delta_h\|_V^2 \leq \alpha_* \, a(\delta_h, \delta_h) \leq a_h(\delta_h, \delta_h) = a_h(p_h, \delta_h) - a_h(p_I, \delta_h)$$

Proof of convergence

Set $\delta_h := p_h - p_I$ ◂ back0

$$\alpha_* \, \alpha \|\delta_h\|_V^2 \leq \alpha_* \, a(\delta_h, \delta_h) \leq a_h(\delta_h, \delta_h) = a_h(p_h, \delta_h) - a_h(p_I, \delta_h)$$
$$= (f_h, \delta_h) - \sum_E a_h^E(p_I, \delta_h)$$

Proof of convergence

Set $\delta_h := p_h - p_I$ ◂ back0

$$\alpha_* \, \alpha \|\delta_h\|_V^2 \le \alpha_* \, a(\delta_h, \delta_h) \le a_h(\delta_h, \delta_h) = a_h(p_h, \delta_h) - a_h(p_I, \delta_h)$$

$$= (f_h, \delta_h) - \sum_E a_h^E(p_I, \delta_h)$$

$$= (f_h, \delta_h) - \sum_E \Big(a_h^E(p_I - p_\pi, \delta_h) + a_h^E(p_\pi, \delta_h) \Big)$$

Proof of convergence

Set $\delta_h := p_h - p_I$ ◂ back0

$$\alpha_* \, \alpha \|\delta_h\|_V^2 \leq \alpha_* \, a(\delta_h, \delta_h) \leq a_h(\delta_h, \delta_h) = a_h(p_h, \delta_h) - a_h(p_I, \delta_h)$$

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Proof of convergence

Set $\delta_h := p_h - p_I$

back0

$$\alpha_* \, \alpha \|\delta_h\|_V^2 \le \alpha_* \, a(\delta_h, \delta_h) \le a_h(\delta_h, \delta_h) = a_h(p_h, \delta_h) - a_h(p_I, \delta_h)$$

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$$= (f_h, \delta_h) - \sum_E \Big(a_h^E(p_I - p_\pi, \delta_h) + a^E(p_\pi, \delta_h) \Big)$$

$$= (f_h, \delta_h) - \sum_E \Big(a_h^E(p_I - p_\pi, \delta_h) + a^E(p_\pi - p, \delta_h) \Big) - a(p, \delta_h)$$

Proof of convergence

Set $\delta_h := p_h - p_I$

back0

$$\alpha_* \, \alpha \|\delta_h\|_V^2 \le \alpha_* \, a(\delta_h, \delta_h) \le a_h(\delta_h, \delta_h) = a_h(p_h, \delta_h) - a_h(p_I, \delta_h)$$

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$$= (f_h, \delta_h) - \sum_E \Big(a_h^E(p_I - p_\pi, \delta_h) + a^E(p_\pi - p, \delta_h) \Big) - a(p, \delta_h)$$

$$= (f_h, \delta_h) - \sum_E \Big(a_h^E(p_I - p_\pi, \delta_h) + a^E(p_\pi - p, \delta_h) \Big) - (f, \delta_h)$$

Proof of convergence

Set $\delta_h := p_h - p_I$ ◂ back0

$$\alpha_* \alpha \|\delta_h\|_V^2 \le \alpha_* a(\delta_h, \delta_h) \le a_h(\delta_h, \delta_h) = a_h(p_h, \delta_h) - a_h(p_I, \delta_h)$$

$$= (f_h, \delta_h) - \sum_E a_h^E(p_I, \delta_h)$$

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$$= (f_h, \delta_h) - \sum_E \Big(a_h^E(p_I - p_\pi, \delta_h) + a^E(p_\pi - p, \delta_h) \Big) - a(p, \delta_h)$$

$$= (f_h, \delta_h) - \sum_E \Big(a_h^E(p_I - p_\pi, \delta_h) + a^E(p_\pi - p, \delta_h) \Big) - (f, \delta_h)$$

$$= (f_h - f, \delta_h) - \sum_E \Big(a_h^E(p_I - p_\pi, \delta_h) + a^E(p_\pi - u, \delta_h) \Big).$$

Consistency and Stability

$$a_h(p_h, q_h) = a(\Pi_k^\nabla p_h, \Pi_k^\nabla q_h) + \mathcal{S}((I - \Pi_k^\nabla)p_h, (I - \Pi_k^\nabla)q_h)$$

Consistency:

$$a_h^E(p_k, q_h) = a^E(p_k, \Pi_k^\nabla q_h) = a^E(\Pi_k^\nabla q_h, p_k) = a^E(p_k, q_h)$$

back1

Consistency and Stability

$$a_h(p_h, q_h) = a(\Pi_k^\nabla p_h, \Pi_k^\nabla q_h) + \mathcal{S}((I - \Pi_k^\nabla)p_h, (I - \Pi_k^\nabla)q_h)$$

Consistency:

$$a_h^E(p_k, q_h) = a^E(p_k, \Pi_k^\nabla q_h) = a^E(\Pi_k^\nabla q_h, p_k) = a^E(p_k, q_h)$$

Stability:

$$a_h^E(q_h, q_h) \le a^E(\Pi_k^\nabla q_h, \Pi_k^\nabla q_h) + c_1 a^E(q_h - \Pi_k^\nabla q_h, q_h - \Pi_k^\nabla q_h)$$
$$= a^E(q_h, \Pi_k^\nabla q_h) + c_1 a^E(q_h - \Pi_k^\nabla q_h, q_h) \le \alpha^* a^E(q_h, q_h)$$

back1

Consistency and Stability

$$a_h(p_h, q_h) = a(\Pi_k^\nabla p_h, \Pi_k^\nabla q_h) + \mathcal{S}((I - \Pi_k^\nabla)p_h, (I - \Pi_k^\nabla)q_h)$$

Consistency:

$$a_h^E(p_k, q_h) = a^E(p_k, \Pi_k^\nabla q_h) = a^E(\Pi_k^\nabla q_h, p_k) = a^E(p_k, q_h)$$

Stability:

$$a_h^E(q_h, q_h) \leq a^E(\Pi_k^\nabla q_h, \Pi_k^\nabla q_h) + c_1 a^E(q_h - \Pi_k^\nabla q_h, q_h - \Pi_k^\nabla q_h)$$
$$= a^E(q_h, \Pi_k^\nabla q_h) + c_1 a^E(q_h - \Pi_k^\nabla q_h, q_h) \leq \alpha^* a^E(q_h, q_h)$$

$$a_h^E(q_h, q_h) \geq a^E(\Pi_k^\nabla q_h, \Pi_k^\nabla q_h) + c_0 a^E(q_h - \Pi_k^\nabla q_h, q_h - \Pi_k^\nabla q_h)$$
$$\geq \alpha_*(a^E(q_h, \Pi_k^\nabla q_h) + a^E(q_h - \Pi_k^\nabla q_h, q_h)) = \alpha_* a^E(q_h, q_h)$$

back1

Proof of unisolvence

Proof. On each element, let ℓ be the number of edges. Then,

$$\begin{aligned} dim\, V_{h|E} &= (k+1)\ell + (\frac{k(k+1)}{2} - 1) + (\frac{k(k+1)}{2}) \\ &= (k+1)\ell + (k^2 + k - 1) \equiv \#d.o.f. \end{aligned}$$

◂ back2

Proof of unisolvence

Proof. On each element, let ℓ be the number of edges. Then,

$$
\begin{aligned}
dim\, V_{h|E} &= (k+1)\ell + (\frac{k(k+1)}{2} - 1) + (\frac{k(k+1)}{2}) \\
&= (k+1)\ell + (k^2 + k - 1) \equiv \#d.o.f.
\end{aligned}
$$

Let $\mathbf{v} \in V_{h|E}$ verify

$$
\int_e \mathbf{v}\cdot\mathbf{n}\, q_k\, d\ell = 0 \qquad \int_E \mathbf{v}\cdot\mathbf{grad}\, q_{k-1}\, dE = 0 \qquad \int_E \mathrm{rot}\,\mathbf{v}\, q_{k-1}\, dE = 0
$$

back2

Proof of unisolvence

Proof. On each element, let ℓ be the number of edges. Then,

$$dim\, V_{h|E} = (k+1)\ell + (\frac{k(k+1)}{2} - 1) + (\frac{k(k+1)}{2})$$
$$= (k+1)\ell + (k^2 + k - 1) \equiv \#d.o.f.$$

Let $\mathbf{v} \in V_{h|E}$ verify

$$\int_e \mathbf{v} \cdot \mathbf{n}\, q_k\, d\ell = 0 \quad \int_E \mathbf{v} \cdot \mathbf{grad} q_{k-1}\, dE = 0 \quad \int_E \mathrm{rot}\, \mathbf{v}\, q_{k-1}\, dE = 0$$

Then, $\mathbf{v} \cdot \mathbf{n}_{|e} \equiv 0,\ \mathrm{div}\, \mathbf{v}_{|E} \equiv 0,\ rot\, \mathbf{v}_{|E} \equiv 0 \implies$

$$\mathbf{v} = \mathbf{curl}\psi, \text{ with } \Delta\psi = 0,\ \frac{\partial\psi}{\partial t} = 0 \implies \psi = const. \implies \mathbf{v} = 0$$

back2

Consistency and stability of $a_h(\cdot,\cdot)$

$$a_h^E(\mathbf{u},\mathbf{v}) := a^E(\widehat{\Pi}^E\mathbf{u},\widehat{\Pi}^E\mathbf{v}) + \mathcal{S}^E((I-\widehat{\Pi}^E)\mathbf{u},(I-\widehat{\Pi}^E)\mathbf{v})$$

back3

Consistency and stability of $a_h(\cdot,\cdot)$

$$a_h^E(\mathbf{u},\mathbf{v}) := a^E(\widehat{\Pi}^E\mathbf{u},\widehat{\Pi}^E\mathbf{v}) + \mathcal{S}^E((I-\widehat{\Pi}^E)\mathbf{u},(I-\widehat{\Pi}^E)\mathbf{v})$$

Consistency: $\forall\, E \in \mathcal{T}_h,\ \forall\, \widehat{\mathbf{u}} \in \widehat{V}^E,\ \forall\, \mathbf{v} \in V_{h|E}$

$$a_h^E(\widehat{\mathbf{u}},\mathbf{v}) = a^E(\widehat{\Pi}^E\widehat{\mathbf{u}},\widehat{\Pi}^E\mathbf{v}) = a^E(\widehat{\Pi}^E\widehat{\mathbf{u}},\mathbf{v}) = a^E(\widehat{\mathbf{u}},\mathbf{v})$$

back3

Consistency and stability of $a_h(\cdot,\cdot)$

$$a_h^E(\mathbf{u},\mathbf{v}) := a^E(\widehat{\Pi}^E\mathbf{u},\widehat{\Pi}^E\mathbf{v}) + \mathcal{S}^E((I-\widehat{\Pi}^E)\mathbf{u},(I-\widehat{\Pi}^E)\mathbf{v})$$

Consistency: $\forall\, E \in \mathcal{T}_h,\ \forall\,\widehat{\mathbf{u}} \in \widehat{V}^E,\ \forall\,\mathbf{v} \in V_{h|E}$

$$a_h^E(\widehat{\mathbf{u}},\mathbf{v}) = a^E(\widehat{\Pi}^E\widehat{\mathbf{u}},\widehat{\Pi}^E\mathbf{v}) = a^E(\widehat{\Pi}^E\widehat{\mathbf{u}},\mathbf{v}) = a^E(\widehat{\mathbf{u}},\mathbf{v})$$

Stability: $\forall\, E \in \mathcal{T}_h,\ \forall\,\mathbf{v} \in V_{h|E}$

$$a_h^E(\mathbf{v},\mathbf{v}) \le a^E(\widehat{\Pi}^E\mathbf{v},\widehat{\Pi}^E\mathbf{v}) + c_1 a^E((I-\widehat{\Pi}^E)\mathbf{v},(I-\widehat{\Pi}^E)\mathbf{v})$$
$$\le \max\{1,c_1\}(a^E(\widehat{\Pi}^E\mathbf{v},\widehat{\Pi}^E\mathbf{v}) + a^E((I-\widehat{\Pi}^E)\mathbf{v},(I-\widehat{\Pi}^E)\mathbf{v}) = \alpha^* a^E(\mathbf{v},\mathbf{v})$$

back3

Consistency and stability of $a_h(\cdot,\cdot)$

$$a_h^E(\mathbf{u},\mathbf{v}) := a^E(\widehat{\Pi}^E\mathbf{u},\widehat{\Pi}^E\mathbf{v}) + \mathcal{S}^E((I-\widehat{\Pi}^E)\mathbf{u},(I-\widehat{\Pi}^E)\mathbf{v})$$

Consistency: $\forall E \in \mathcal{T}_h,\ \forall \widehat{\mathbf{u}} \in \widehat{V}^E,\ \forall \mathbf{v} \in V_{h|E}$

$$a_h^E(\widehat{\mathbf{u}},\mathbf{v}) = a^E(\widehat{\Pi}^E\widehat{\mathbf{u}},\widehat{\Pi}^E\mathbf{v}) = a^E(\widehat{\Pi}^E\widehat{\mathbf{u}},\mathbf{v}) = a^E(\widehat{\mathbf{u}},\mathbf{v})$$

Stability: $\forall E \in \mathcal{T}_h,\ \forall \mathbf{v} \in V_{h|E}$

$$a_h^E(\mathbf{v},\mathbf{v}) \le a^E(\widehat{\Pi}^E\mathbf{v},\widehat{\Pi}^E\mathbf{v}) + c_1 a^E((I-\widehat{\Pi}^E)\mathbf{v},(I-\widehat{\Pi}^E)\mathbf{v})$$
$$\le \max\{1,c_1\}(a^E(\widehat{\Pi}^E\mathbf{v},\widehat{\Pi}^E\mathbf{v}) + a^E((I-\widehat{\Pi}^E)\mathbf{v},(I-\widehat{\Pi}^E)\mathbf{v}) = \alpha^* a^E(\mathbf{v},\mathbf{v})$$

$$a_h^E(\mathbf{v},\mathbf{v}) \ge a^E(\widehat{\Pi}^E\mathbf{v},\widehat{\Pi}^E\mathbf{v}) + c_0 a^E((I-\widehat{\Pi}^E)\mathbf{v},(I-\widehat{\Pi}^E)\mathbf{v})$$
$$\ge \min\{1,c_0\}(a^E(\widehat{\Pi}^E\mathbf{v},\widehat{\Pi}^E\mathbf{v}) + a^E((I-\widehat{\Pi}^E)\mathbf{v},(I-\widehat{\Pi}^E)\mathbf{v}) = \alpha_* a^E(\mathbf{v},\mathbf{v})$$

back3

Proof of convergence

Set $\boldsymbol{\delta}_h = \mathbf{u}_h - \mathbf{u}_I$ and notice that $\operatorname{div} \boldsymbol{\delta}_h = 0$. Hence, $a_h(\mathbf{u}_h, \boldsymbol{\delta}_h) = 0$.

back4

Proof of convergence

Set $\boldsymbol{\delta}_h = \mathbf{u}_h - \mathbf{u}_I$ and notice that $\mathrm{div}\,\boldsymbol{\delta}_h = 0$. Hence, $a_h(\mathbf{u}_h, \boldsymbol{\delta}_h) = 0$.

$$\alpha_* \alpha \|\boldsymbol{\delta}_h\|^2_{L^2} \leq \alpha_* a(\boldsymbol{\delta}_h, \boldsymbol{\delta}_h) \leq a_h(\boldsymbol{\delta}_h, \boldsymbol{\delta}_h)$$

back4

Proof of convergence

Set $\boldsymbol{\delta}_h = \mathbf{u}_h - \mathbf{u}_I$ and notice that $\operatorname{div} \boldsymbol{\delta}_h = 0$. Hence, $a_h(\mathbf{u}_h, \boldsymbol{\delta}_h) = 0$.

$$\alpha_* \alpha \|\boldsymbol{\delta}_h\|^2_{L^2} \leq \alpha_* a(\boldsymbol{\delta}_h, \boldsymbol{\delta}_h) \leq a_h(\boldsymbol{\delta}_h, \boldsymbol{\delta}_h)$$
$$= a_h(\mathbf{u}_h, \boldsymbol{\delta}_h) - a_h(\mathbf{u}_I, \boldsymbol{\delta}_h) = -\sum_E a_h^E(\mathbf{u}_I, \boldsymbol{\delta}_h)$$

back4

Proof of convergence

Set $\boldsymbol{\delta}_h = \mathbf{u}_h - \mathbf{u}_I$ and notice that $\mathrm{div}\,\boldsymbol{\delta}_h = 0$. Hence, $a_h(\mathbf{u}_h, \boldsymbol{\delta}_h) = 0$.

$$
\begin{aligned}
\alpha_* \alpha \|\boldsymbol{\delta}_h\|^2_{L^2} \le \alpha_* a(\boldsymbol{\delta}_h, \boldsymbol{\delta}_h) \le a_h(\boldsymbol{\delta}_h, \boldsymbol{\delta}_h) \\
= a_h(\mathbf{u}_h, \boldsymbol{\delta}_h) - a_h(\mathbf{u}_I, \boldsymbol{\delta}_h) = -\sum_E a_h^E(\mathbf{u}_I, \boldsymbol{\delta}_h) \\
= -\sum_E \left(a_h^E(\mathbf{u}_I - \mathbf{u}_\pi, \boldsymbol{\delta}_h) + a_h^E(\mathbf{u}_\pi, \boldsymbol{\delta}_h) \right)
\end{aligned}
$$

back4

Proof of convergence

Set $\boldsymbol{\delta}_h = \mathbf{u}_h - \mathbf{u}_I$ and notice that $\operatorname{div} \boldsymbol{\delta}_h = 0$. Hence, $a_h(\mathbf{u}_h, \boldsymbol{\delta}_h) = 0$.

$$\begin{aligned}
\alpha_* \alpha \|\boldsymbol{\delta}_h\|^2_{L^2} \le \alpha_* a(\boldsymbol{\delta}_h, \boldsymbol{\delta}_h) &\le a_h(\boldsymbol{\delta}_h, \boldsymbol{\delta}_h) \\
&= a_h(\mathbf{u}_h, \boldsymbol{\delta}_h) - a_h(\mathbf{u}_I, \boldsymbol{\delta}_h) = -\sum_E a_h^E(\mathbf{u}_I, \boldsymbol{\delta}_h) \\
&= -\sum_E \left(a_h^E(\mathbf{u}_I - \mathbf{u}_\pi, \boldsymbol{\delta}_h) + a_h^E(\mathbf{u}_\pi, \boldsymbol{\delta}_h) \right) \\
&= -\sum_E \left(a_h^E(\mathbf{u}_I - \mathbf{u}_\pi, \boldsymbol{\delta}_h) + a^E(\mathbf{u}_\pi, \boldsymbol{\delta}_h) \right)
\end{aligned}$$

back4

Proof of convergence

Set $\boldsymbol{\delta}_h = \mathbf{u}_h - \mathbf{u}_I$ and notice that $\operatorname{div} \boldsymbol{\delta}_h = 0$. Hence, $a_h(\mathbf{u}_h, \boldsymbol{\delta}_h) = 0$.

$$
\begin{aligned}
\alpha_* \alpha \|\boldsymbol{\delta}_h\|^2_{L^2} \le \alpha_* a(\boldsymbol{\delta}_h, \boldsymbol{\delta}_h) &\le a_h(\boldsymbol{\delta}_h, \boldsymbol{\delta}_h) \\
&= a_h(\mathbf{u}_h, \boldsymbol{\delta}_h) - a_h(\mathbf{u}_I, \boldsymbol{\delta}_h) = -\sum_E a_h^E(\mathbf{u}_I, \boldsymbol{\delta}_h) \\
&= -\sum_E \Big(a_h^E(\mathbf{u}_I - \mathbf{u}_\pi, \boldsymbol{\delta}_h) + a_h^E(\mathbf{u}_\pi, \boldsymbol{\delta}_h) \Big) \\
&= -\sum_E \Big(a_h^E(\mathbf{u}_I - \mathbf{u}_\pi, \boldsymbol{\delta}_h) + a^E(\mathbf{u}_\pi, \boldsymbol{\delta}_h) \Big) \\
&= -\sum_E \Big(a_h^E(\mathbf{u}_I - \mathbf{u}_\pi, \boldsymbol{\delta}_h) + a^E(\mathbf{u}_\pi - \mathbf{u}, \boldsymbol{\delta}_h) \Big) - a(\mathbf{u}, \boldsymbol{\delta}_h)
\end{aligned}
$$

back4

Proof of convergence

Set $\boldsymbol{\delta}_h = \mathbf{u}_h - \mathbf{u}_I$ and notice that $\operatorname{div} \boldsymbol{\delta}_h = 0$. Hence, $a_h(\mathbf{u}_h, \boldsymbol{\delta}_h) = 0$.

$$\alpha_* \alpha \|\boldsymbol{\delta}_h\|^2_{L^2} \le \alpha_* a(\boldsymbol{\delta}_h, \boldsymbol{\delta}_h) \le a_h(\boldsymbol{\delta}_h, \boldsymbol{\delta}_h)$$

$$= a_h(\mathbf{u}_h, \boldsymbol{\delta}_h) - a_h(\mathbf{u}_I, \boldsymbol{\delta}_h) = -\sum_E a_h^E(\mathbf{u}_I, \boldsymbol{\delta}_h)$$

$$= -\sum_E \Big(a_h^E(\mathbf{u}_I - \mathbf{u}_\pi, \boldsymbol{\delta}_h) + a_h^E(\mathbf{u}_\pi, \boldsymbol{\delta}_h) \Big)$$

$$= -\sum_E \Big(a_h^E(\mathbf{u}_I - \mathbf{u}_\pi, \boldsymbol{\delta}_h) + a^E(\mathbf{u}_\pi, \boldsymbol{\delta}_h) \Big)$$

$$= -\sum_E \Big(a_h^E(\mathbf{u}_I - \mathbf{u}_\pi, \boldsymbol{\delta}_h) + a^E(\mathbf{u}_\pi - \mathbf{u}, \boldsymbol{\delta}_h) \Big) - a(\mathbf{u}, \boldsymbol{\delta}_h)$$

$$= -\sum_E \Big(a_h^E(\mathbf{u}_I - \mathbf{u}_\pi, \boldsymbol{\delta}_h) + a^E(\mathbf{u}_\pi - \mathbf{u}, \boldsymbol{\delta}_h) \Big).$$

back4