The Multi-level Monte Carlo Technique for Approximation of Distribution Functions

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I. The Computational Problem

Given a random variable τ, **determine** the distribution function F of τ,

$$F(s) = P(\{\tau \leq s\}).$$

Example Hitting time τ of a stochastic process X.

In this talk Monte Carlo algorithms for approximation of F.

Alternatives include analytic formulas and numerics of PDEs.

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Example Hitting time $\tau^{(\ell)}$ of an approximation $X^{(\ell)}$ of X.

Later on, we study approximation of F on compact intervals. Note that

$$F(s) = \mathrm{E}(1_{]-\infty,s]}(\tau)).$$

Classical Monte Carlo Take $L \in \mathbb{N}_0$ and $N \in \mathbb{N}$, and approximate F by

$$s \mapsto \frac{1}{N} \sum_{i=1}^{N} 1_{]-\infty,s]}(\tau_i^{(L)})$$

with independent copies $\tau_1^{(L)}, \ldots, \tau_N^{(L)}$ of $\tau^{(L)}$. Cf. empirical distribution function.

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Basic ideas for improvement

- Smoothing of $1_{]-\infty,s]}$, provided that

  τ has a smooth density.

  Cf. kernel density estimation.

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Basic ideas for improvement

- Smoothing of $1_{]-\infty,s]}$, provided that

  τ has a smooth density.

- Multi-level approach, using coupled simulation of $\tau^{(0)}, \dots, \tau^{(L)}$, provided that

$$\lim_{\ell \to \infty} \mathrm{E}(\tau - \tau^{(\ell)})^2 = 0.$$

II. Monte Carlo Algorithms

Let $f : \mathbb{R} \to \mathbb{R}$, e.g., $f = 1_{]-\infty,s]}$ with s fixed. Compute

$$a = \mathrm{E}(f(\tau)).$$

Monte Carlo algorithm: an algorithm that uses random numbers.

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Clearly

$$\mathrm{error}^2(A) = (\underbrace{a - \mathrm{E}(A)}_{\mathrm{bias}(A)})^2 + \mathrm{Var}(A).$$

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'Definition' A sequence of Monte Carlo algorithms A_n with $\lim_{n\to\infty} \mathrm{cost}(A_n) = \infty$ achieves order of convergence $\gamma > 0$ if

$$\exists\, c > 0\, \exists\, \eta \in \mathbb{R}\, \forall\, n \in \mathbb{N} :$$
$$\mathrm{error}(A_n) \leq c \cdot \big(\mathrm{cost}(A_n)\big)^{-\gamma} \cdot \big(\log \mathrm{cost}(A_n)\big)^{\eta}.$$

III. Single-level MC

Assumptions

(S1) $f : \mathbb{R} \to \mathbb{R}$ and

$$\exists\, c > 0\, \forall\, x \in \mathbb{R} : \quad \mathrm{cost}(f(x)) \leq c.$$

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(S3) There exists $\alpha > 0$ such that

$$\exists\, c > 0\, \forall\, \ell \in \mathbb{N}_0 : \quad \left|\mathrm{E}\left(f(\tau) - f(\tau^{(\ell)})\right)\right| \leq c \cdot M^{-\ell \cdot \alpha}.$$

(S4) $\sup_{\ell \in \mathbb{N}} \mathrm{Var}(f(\tau^{(\ell)})) < \infty$.

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Example $\tau^{(\ell)}$ Euler approximation of an SDE at time T with step-size $T/2^{\ell}$. Under standard assumptions,

$$M = 2, \qquad\qquad \alpha = 1.$$

Single-level Monte Carlo $A_N^L = \frac{1}{N}\sum_{i=1}^{N} f(\tau_i^{(L)})$ yields

$$\begin{aligned}\mathrm{error}^2(A_N^L) &= \big(a - \mathrm{E}(f(\tau^{(L)}))\big)^2 + \frac{1}{N}\,\mathrm{Var}(f(\tau^{(L)}))\\ &\le c \cdot \big(M^{-2\ell\cdot\alpha} + N^{-1}\big),\end{aligned}$$

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Example For SDEs and the Euler approximation

$$\gamma = \frac{1}{3}.$$

More generally, weak approximation of SDEs.

IV. Multi-level MC: The Lipschitz Case

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Assumptions: **(S3)** and

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Example For SDEs and the Euler approximation

$$\beta = 1/2.$$

Clearly

$$\mathrm{E}(f(\tau^{(L)})) = \mathrm{E}(f(\tau^{(0)})) + \sum_{\ell=1}^{L} \mathrm{E}\big(f(\tau^{(\ell)}) - f(\tau^{(\ell-1)})\big).$$

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Idea: variance reduction, compared to single-level MC, by approximating

$$f(\tau^{(0)}), f(\tau^{(1)}) - f(\tau^{(0)}), \ldots, f(\tau^{(L)}) - f(\tau^{(L_1)})$$

separately with independent MC algorithms.

Definition of the multi-level algorithm

Consider an

- independent family of $\mathbb{R}^2$-valued random variables $\left(\tau_i^{(\ell)}, \sigma_i^{(\ell)}\right)$ such that $\left(\tau_i^{(\ell)}, \sigma_i^{(\ell)}\right) \overset{\mathrm{d}}{=} \left(\tau^{(\ell)}, \tau^{(\ell-1)}\right)$. Here $\tau^{(-1)} = 0$, say.

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- minimal and maximal levels $L_0, L_1 \in \mathbb{N}$ and
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Put

$$A^{L_0,L_1}_{N_0,\ldots,N_L} = \underbrace{\frac{1}{N_{L_0}} \cdot \sum_{i=1}^{N_{L_0}} f\big(\tau_i^{(L_0)}\big)}_{\to \mathrm{E}(f(\tau^{(L_0)}))} + \sum_{\ell=L_0+1}^{L_1} \underbrace{\frac{1}{N_\ell} \cdot \sum_{i=1}^{N_\ell} \Big(f\big(\tau_i^{(\ell)}\big) - f\big(\sigma_i^{(\ell)}\big)\Big)}_{\to \mathrm{E}\left(f(\tau^{(\ell)}) - f(\tau^{(\ell-1)})\right)} .$$

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Example For SDEs and the Euler approximation

$$\gamma = \frac{1}{2} \qquad \text{vs.} \qquad \gamma = \frac{1}{3}.$$

Multi-level Monte Carlo

- Integral equations, parametric integration

  Heinrich (1998), Heinrich, Sindambiwe (1999).

  Here f actually takes values in an infinite-dimensional space.

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Here τ and $\tau^{(\ell)}$ may take values in an infinite-dimensional space.

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- Optimality of MLMC algorithms for diffusions or Gaussian processes τ

  Creutzig, Dereich, Müller-Gronbach, R (2009).

  Here worst case analysis on the Lipschitz class.

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Assume that $\tau \geq 0$. We study approximation of

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Stopping: Study approximation of $\tau \wedge T$ by $\tau^{(\ell)}$, where

$$T = S + 1$$

and, by assumption,

$$\tau^{(\ell)} \in [0, T].$$

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For non-smooth functionals of SDEs, see also

Avikainen (2009), *Giles, Higham, Mao (2009)*,

Altmayer, Neuenkirch (2012).

Smoothing: Assumption

(D1) There exists $r \in \mathbb{N}_0$ such that τ has a density $\rho \in C^r([0,\infty[)$ and

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Approximate $1_{[0,s]}$ by rescaled translates

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Example Let Φ denote the standard normal distribution function. For $r = 1$ take

$$g(u) = \Phi(-u).$$

For $r = 3$ take

$$g(u) = 4/3 \cdot \Phi(-u) - 1/3 \cdot \Phi(-u/2).$$

In general, take a bounded Lipschitz function $g : \mathbb{R} \to \mathbb{R}$ such that

$$\exists\, c > 0\, \forall\, s \in \mathbb{R} : \quad \operatorname{cost}(g(s)) \le c,$$

$$\int_{-\infty}^{\infty} |s|^r \cdot |1_{]-\infty,0]}(s) - g(s)|\, ds < \infty,$$

$$\sup_{s \in [1,\infty[} |g(s)| \cdot s^{r+1} < \infty,$$

$$\forall\, j = 0, \dots, r-1 : \quad \int_{-\infty}^{\infty} s^j \cdot (1_{]-\infty,0]}(s) - g(s))\, ds = 0.$$

Assumption: **(D1)**, **(M2)**, and

(D3) There exists $\alpha > 0$ such that

$$\exists\, c > 0 \,\forall\, \delta > 0 \,\forall\, t \in [0, T] \,\forall\, \ell \in \mathbb{N}_0 :$$
$$\left| \mathrm{E} \left(g(\tfrac{\tau \wedge T - t}{\delta}) - g(\tfrac{\tau^{(\ell)} \wedge T - t}{\delta}) \right) \right| \leq c/\delta \cdot M^{-\ell \cdot \alpha}.$$

(D4) There exists $\beta \in \,]0, \alpha]$ such that

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Definition of the multi-level algorithm

Step 1 Approximation of F at

$$s_i = i \cdot S/k, \qquad i = 1, \ldots, k.$$

Replace $f : \mathbb{R} \to \mathbb{R}$ by $g^{k,\delta} : \mathbb{R} \to \mathbb{R}^k$,

$$g^{k,\delta}(t) = \left(g(\tfrac{t-s_1}{\delta}), \ldots, g(\tfrac{t-s_k}{\delta})\right).$$

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with additional parameters $k \in \mathbb{N}$ and $\delta > 0$.

Step 2 Extension to functions on $[0, S]$.

Take linear mappings $P_k : \mathbb{R}^k \to C([0, S])$ such that $\exists\, c > 0\, \forall\, k \in \mathbb{N}$

$$\forall x \in \mathbb{R}^k : \quad \mathrm{cost}(P_k(x)) \leq c \cdot k,$$

$$\forall x \in \mathbb{R}^k : \quad \|P_k(x)\|_\infty \leq c \cdot |x|_\infty,$$

$$\|F - P_k(F(s_1), \ldots, F(s_k))\|_\infty \leq c \cdot k^{-(r+1)}.$$

Example P_k piecewise polynomial interpolation of degree r, taking into account $F(0) = 0$.

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Steps 1 and 2 yield the algorithm

$$\mathcal{M}^{k,\delta,L_0,L_1}_{N_{L_0},\dots,N_{L_1}} = P_k(A^{k,\delta,L_0,L_1}_{N_{L_0},\dots,N_{L_1}}).$$

As previously, for convenience, $\beta \leq 1/2$.

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Theorem *Giles, Iliev, Nagapetyan, R (2012)*

Multi-level Monte Carlo achieves order of convergence

$$\begin{aligned}
q \leq 1 \quad &\Rightarrow \quad \gamma = \frac{r+1}{2r+3}, \\
(1 < q \leq 2) \vee \left(q > 2 \wedge \beta \leq \frac{1}{q}\right) \quad &\Rightarrow \quad \gamma = \frac{r+1}{2(r+1)+q}, \\
q > 2 \wedge \beta > \frac{1}{q} \quad &\Rightarrow \quad \gamma = \frac{r+1}{r+2} \cdot \frac{\alpha}{1+2(\alpha-\beta)}.
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$$q > 2 \wedge \beta > \frac{1}{q} \quad \Rightarrow \quad \gamma = \frac{r+1}{r+2} \cdot \frac{\alpha}{1+2(\alpha-\beta)}.$$

In the first two cases the orders are actually achieved by single-level MC,

i.e., with $L_0 = L_1$.

Proof: We have

$$\mathrm{error}^2(Q_k(\mathcal{M})) \preceq k^{-2(r+1)} + \delta^{2(r+1)} + 1/\delta^2 \cdot M^{-2L_1 \cdot \alpha}$$
$$+ \log k \cdot \left(\frac{1}{N_{L_0}} + \sum_{\ell=L_0+1}^{L_1} \frac{M^{-2\ell \cdot \beta}}{N_\ell \cdot \delta^2} \right)$$

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Hence, for k fixed and $\delta = 1/k$,

$$k \leq M^{L_0} \leq M^{L_1} = k^{\max(1,q)}.$$

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Hence, for k fixed and $\delta = 1/k$,

$$k \leq M^{L_0} \leq M^{L_1} = k^{\max(1,q)}.$$

Furthermore, if $q > 1$,

$$\frac{M^{-2L_1\cdot\beta}}{\delta^2} = k^{2(1-\beta q)}.$$

VI. SDEs with Reflection, AF^4

- Separation of nano-particles of different types, domain $D \subset \mathbb{R}^d$.
- Ignoring interactions, the motion of a particle is described by an SDE

$$dX(t) = \mu(X(t))\,dt + \sigma(X(t))\,dW(t) + d\phi(t)$$

  with normal reflection on ∂D.
- Instead of reflection, we study absorption at $\partial_{\mathrm{a}} D \subset \partial D$ and consider the hitting time

$$\tau = \inf\{t \geq 0 : X(t) \in \partial_{\mathrm{a}} D\}.$$

See

Gobet, Menozzi (2010), *Higham, Mao, Roy, Song, Yin* et al. *(2011)*, *Słomiński (2001)*, *Costantini, Pacchiarotti, Satoretto (1998)*, *Bayer, Szepessy, Tempone (2010)*.